1.1 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.

Exercises 1–5, decide whether the problem can be solved using ecalculus or whether calculus is required. If the problem can be ved using precalculus, solve it. If the problem seems to require lculus, explain your reasoning and use a graphical or numerical proach to estimate the solution.

Find the distance traveled in 15 seconds by an object traveling at a constant velocity of 20 feet per second.

Find the distance traveled in 15 seconds by an object moving with a velocity of $v(t) = 20 + 7\cos t$ feet per second.

A bicyclist is riding on a path modeled by the function $f(x) = 0.04(8x - x^2)$, where x and $f(x)$ are measured in miles. Find the rate of change of elevation at $x = 2$.

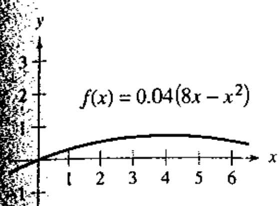


Figure for 3

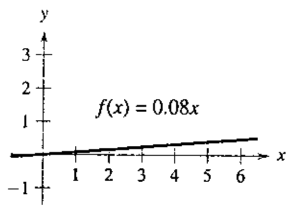
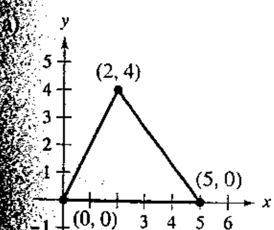


Figure for 4

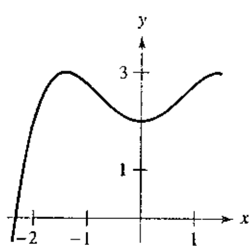
A bicyclist is riding on a path modeled by the function $f(x) = 0.08x$, where x and $f(x)$ are measured in miles. Find the rate of change of elevation at $x = 2$.

Find the area of the shaded region.

(a)



(b)



Secant Lines Consider the function $f(x) = \sqrt{x}$ and the point $(4, 2)$ on the graph of f.

(a) Graph f and the secant lines passing through $P(4, 2)$ and $Q(x, f(x))$ for x-values of 1, 3, and 5.

(b) Find the slope of each secant line.

(c) Use the results of part (b) to estimate the slope of the tangent line to the graph of f at $P(4, 2)$. Describe how to improve your approximation of the slope.

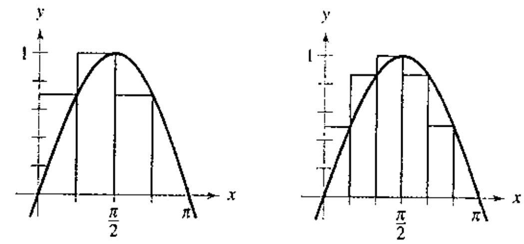
Secant Lines Consider the function $f(x) = 6x - x^2$ and the point $P(2, 8)$ on the graph of f.

(a) Graph f and the secant lines passing through $P(2, 8)$ and $Q(x, f(x))$ for x-values of 3, 2.5, and 1.5.

(b) Find the slope of each secant line.

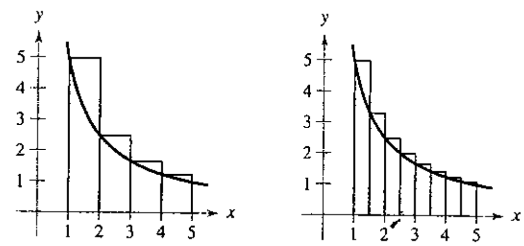
(c) Use the results of part (b) to estimate the slope of the tangent line to the graph of f at $P(2, 8)$. Describe how to improve your approximation of the slope.

8. (a) Use the rectangles in each graph to approximate the area of the region bounded by $y = \sin x$, $y = 0$, $x = 0$, and $x = \pi$.



(b) Describe how you could continue this process to obtain a more accurate approximation of the area.

9. (a) Use the rectangles in each graph to approximate the area of the region bounded by $y = 5/x$, $y = 0$, $x = 1$, and $x = 5$.



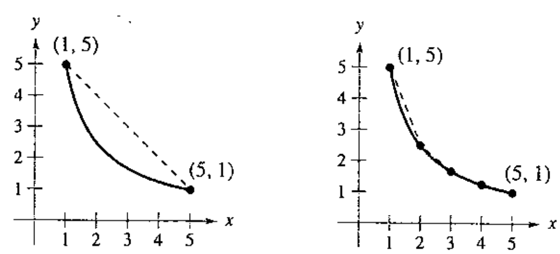
(b) Describe how you could continue this process to obtain a more accurate approximation of the area.

CAPSTONE

10. How would you describe the instantaneous rate of change of an automobile's position on the highway?

WRITING ABOUT CONCEPTS

11. Consider the length of the graph of $f(x) = 5/x$ from $(1, 5)$ to $(5, 1)$.



(a) Approximate the length of the curve by finding the distance between its two endpoints, as shown in the first figure.

(b) Approximate the length of the curve by finding the sum of the lengths of four line segments, as shown in the second figure.

(c) Describe how you could continue this process to obtain a more accurate approximation of the length of the curve.

In Exercises 9–14, create a table of values for the function and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.

9. $\lim\limits_{x\to 1} \dfrac{x-2}{x^2+x-6}$

10. $\lim\limits_{x\to -3} \dfrac{x+3}{x^2+7x+12}$

11. $\lim\limits_{x\to 1} \dfrac{x^4-1}{x^6-1}$

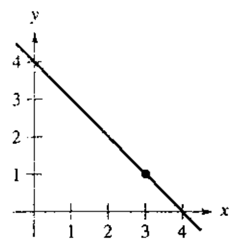
12. $\lim\limits_{x\to -2} \dfrac{x^3+8}{x+2}$

13. $\lim\limits_{x\to 0} \dfrac{\sin 2x}{x}$

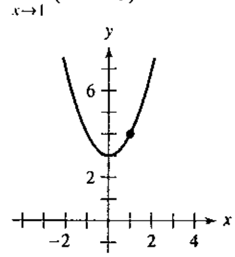
14. $\lim\limits_{x\to 0} \dfrac{\tan x}{\tan 2x}$

In Exercises 15–24, use the graph to find the limit (if it exists). If the limit does not exist, explain why.

15. $\lim\limits_{x\to 3} (4-x)$

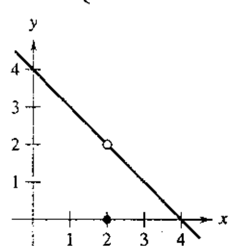


16. $\lim\limits_{x\to 1} (x^2+3)$



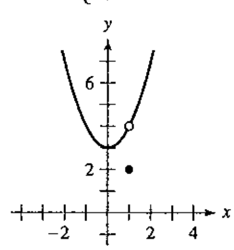
17. $\lim\limits_{x\to 2} f(x)$

$$f(x) = \begin{cases} 4-x, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

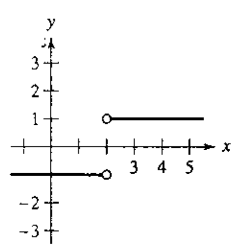


18. $\lim\limits_{x\to 1} f(x)$

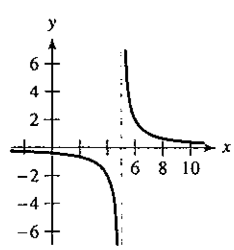
$$f(x) = \begin{cases} x^2+3, & x \neq 1 \\ 2, & x = 1 \end{cases}$$



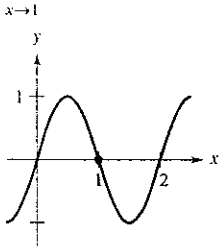
19. $\lim\limits_{x\to 2} \dfrac{|x-2|}{x-2}$



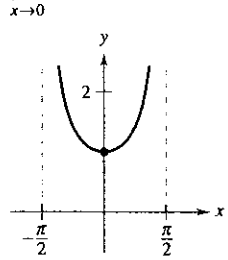
20. $\lim\limits_{x\to 5} \dfrac{2}{x-5}$



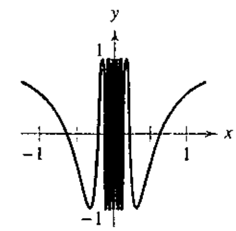
21. $\lim\limits_{x\to 1} \sin \pi x$



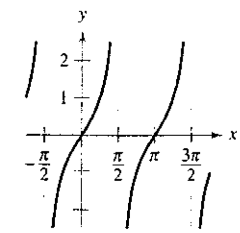
22. $\lim\limits_{x\to 0} \sec x$



23. $\lim\limits_{x\to 0} \cos \dfrac{1}{x}$



24. $\lim\limits_{x\to \pi/2} \tan x$



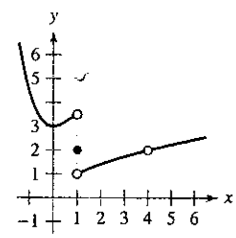
In Exercises 25 and 26, use the graph of the function f to decide whether the value of the given quantity exists. If it does, find it. If not, explain why.

25. (a) $f(1)$

(b) $\lim\limits_{x\to 1} f(x)$

(c) $f(4)$

(d) $\lim\limits_{x\to 4} f(x)$



26. (a) $f(-2)$

(b) $\lim\limits_{x\to -2} f(x)$

(c) $f(0)$

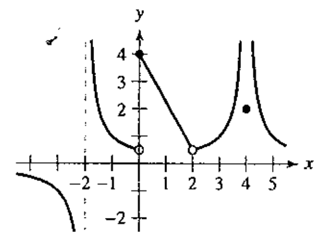
(d) $\lim\limits_{x\to 0} f(x)$

(e) $f(2)$

(f) $\lim\limits_{x\to 2} f(x)$

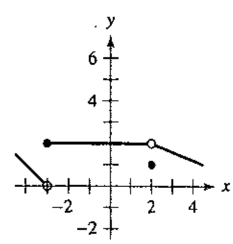
(g) $f(4)$

(h) $\lim\limits_{x\to 4} f(x)$

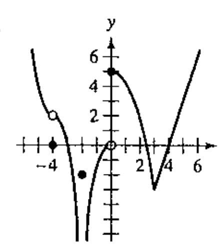


In Exercises 27 and 28, use the graph of f to identify the values of c for which $\lim\limits_{x\to c} f(x)$ exists.

27.



28.



In Exercises 29 and 30, sketch the graph of f. Then identify the values of c for which $\lim\limits_{x\to c} f(x)$ exists.

29. $f(x) = \begin{cases} x^2, & x \le 2 \\ 8-2x, & 2 < x < 4 \\ 4, & x \ge 4 \end{cases}$

30. $f(x) = \begin{cases} \sin x, & x < 0 \\ 1-\cos x, & 0 \le x \le \pi \\ \cos x, & x > \pi \end{cases}$

EXAMPLE 8 Using the ε-δ Definition of Limit

Use the ε-δ definition of limit to prove that

$$\lim_{x \to 2} x^2 = 4.$$

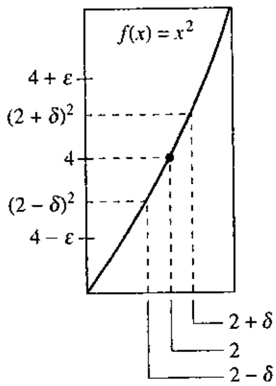
Solution You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x^2 - 4| < \varepsilon \text{ whenever } 0 < |x - 2| < \delta.$$

To find an appropriate δ, begin by writing $|x^2 - 4| = |x - 2||x + 2|$. For all x in th interval $(1, 3)$, $x + 2 < 5$ and thus $|x + 2| < 5$. So, letting δ be the minimum of $\varepsilon/$ and 1, it follows that, whenever $0 < |x - 2| < \delta$, you have

$$|x^2 - 4| = |x - 2||x + 2| < \left(\frac{\varepsilon}{5}\right)(5) = \varepsilon$$

as shown in Figure 1.15.



The limit of $f(x)$ as x approaches 2 is 4.
Figure 1.15

Throughout this chapter you will use the ε-δ definition of limit primarily to pro theorems about limits and to establish the existence or nonexistence of particular typ of limits. For *finding* limits, you will learn techniques that are easier to use than the ε- definition of limit.

1.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.

1. $\displaystyle\lim_{x \to 4} \frac{x - 4}{x^2 - 3x - 4}$

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$						

2. $\displaystyle\lim_{x \to 2} \frac{x - 2}{x^2 - 4}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

3. $\displaystyle\lim_{x \to 0} \frac{\sqrt{x + 6} - \sqrt{6}}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

4. $\displaystyle\lim_{x \to -5} \frac{\sqrt{4 - x} - 3}{x + 5}$

x	-5.1	-5.01	-5.001	-4.999	-4.99	-4.9
$f(x)$						

5. $\displaystyle\lim_{x \to 3} \frac{[1/(x + 1)] - (1/4)}{x - 3}$

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$						

6. $\displaystyle\lim_{x \to 4} \frac{[x/(x + 1)] - (4/5)}{x - 4}$

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$						

7. $\displaystyle\lim_{x \to 0} \frac{\sin x}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

8. $\displaystyle\lim_{x \to 0} \frac{\cos x - 1}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

70. ***Graphical Analysis*** The statement

$$\lim_{x\to 3} \frac{x^2 - 3x}{x - 3}$$

means that for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that if $0 < |x - 3| < \delta$, then

$$\left|\frac{x^2 - 3x}{x - 3} - 3\right| < \varepsilon.$$

If $\varepsilon = 0.001$, then

$$\left|\frac{x^2 - 3x}{x - 3} - 3\right| < 0.001.$$

Use a graphing utility to graph each side of this inequality. Use the *zoom* feature to find an interval $(3 - \delta, 3 + \delta)$ such that the graph of the left side is below the graph of the right side of the inequality.

True or False? **In Exercises 71–74, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

71. If f is undefined at $x = c$, then the limit of $f(x)$ as x approaches c does not exist.

72. If the limit of $f(x)$ as x approaches c is 0, then there must exist a number k such that $f(k) < 0.001$.

73. If $f(c) = L$, then $\lim_{x\to c} f(x) = L$.

74. If $\lim_{x\to c} f(x) = L$, then $f(c) = L$.

In Exercises 75 and 76, consider the function $f(x) = \sqrt{x}$.

75. Is $\lim_{x\to 0.25} \sqrt{x} = 0.5$ a true statement? Explain.

76. Is $\lim_{x\to 0} \sqrt{x} = 0$ a true statement? Explain.

77. Use a graphing utility to evaluate the limit $\lim_{x\to 0} \dfrac{\sin nx}{x}$ for several values of n. What do you notice?

78. Use a graphing utility to evaluate the limit $\lim_{x\to 0} \dfrac{\tan nx}{x}$ for several values of n. What do you notice?

79. Prove that if the limit of $f(x)$ as $x \to c$ exists, then the limit must be unique. [*Hint:* Let

$$\lim_{x\to c} f(x) = L_1 \quad \text{and} \quad \lim_{x\to c} f(x) = L_2$$

and prove that $L_1 = L_2$.]

80. Consider the line $f(x) = mx + b$, where $m \neq 0$. Use the ε-δ definition of limit to prove that $\lim_{x\to c} f(x) = mc + b$.

81. Prove that $\lim_{x\to c} f(x) = L$ is equivalent to $\lim_{x\to c} [f(x) - L] = 0$.

82. (a) Given that

$$\lim_{x\to 0} (3x + 1)(3x - 1)x^2 + 0.01 = 0.01$$

prove that there exists an open interval (a, b) containing 0 such that $(3x + 1)(3x - 1)x^2 + 0.01 > 0$ for all $x \neq 0$ in (a, b).

(b) Given that $\lim_{x\to c} g(x) = L$, where $L > 0$, prove that there exists an open interval (a, b) containing c such that $g(x) > 0$ for all $x \neq c$ in (a, b).

83. ***Programming*** Use the programming capabilities of a graphing utility to write a program for approximating $\lim_{x\to c} f(x)$.

Assume the program will be applied only to functions whose limits exist as x approaches c. Let $y_1 = f(x)$ and generate two lists whose entries form the ordered pairs

$$(c \pm [0.1]^n,\ f(c \pm [0.1]^n))$$

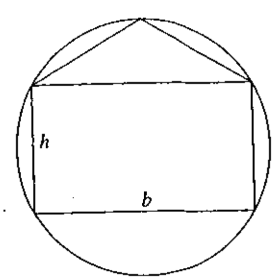
for $n = 0, 1, 2, 3,$ and 4.

84. ***Programming*** Use the program you created in Exercise 83 to approximate the limit

$$\lim_{x\to 4} \frac{x^2 - x - 12}{x - 4}.$$

PUTNAM EXAM CHALLENGE

85. Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown. For what value of h do the rectangle and triangle have the same area?



86. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

1.3 Exercises

In Exercises 1–4, use a graphing utility to graph the function and visually estimate the limits.

1. $h(x) = -x^2 + 4x$
   (a) $\lim_{x\to 4} h(x)$
   (b) $\lim_{x\to -1} h(x)$

2. $g(x) = \dfrac{12(\sqrt{x} - 3)}{x - 9}$
   (a) $\lim_{x\to 4} g(x)$
   (b) $\lim_{x\to 0} g(x)$

3. $f(x) = x \cos x$
   (a) $\lim_{x\to 0} f(x)$
   (b) $\lim_{x\to \pi/3} f(x)$

4. $f(t) = t|t - 4|$
   (a) $\lim_{t\to 4} f(t)$
   (b) $\lim_{t\to -1} f(t)$

In Exercises 5–22, find the limit.

5. $\lim_{x\to 2} x^3$
6. $\lim_{x\to -2} x^4$
7. $\lim_{x\to 0} (2x - 1)$
8. $\lim_{x\to -3} (3x + 2)$
9. $\lim_{x\to -3} (x^2 + 3x)$
10. $\lim_{x\to 1} (-x^2 + 1)$
11. $\lim_{x\to -3} (2x^2 + 4x + 1)$
12. $\lim_{x\to 1} (3x^3 - 2x^2 + 4)$
13. $\lim_{x\to 3} \sqrt{x + 1}$
14. $\lim_{x\to 4} \sqrt[3]{x + 4}$
15. $\lim_{x\to -4} (x + 3)^2$
16. $\lim_{x\to 0} (2x - 1)^3$
17. $\lim_{x\to 2} \dfrac{1}{x}$
18. $\lim_{x\to -3} \dfrac{2}{x + 2}$
19. $\lim_{x\to 1} \dfrac{x}{x^2 + 4}$
20. $\lim_{x\to 1} \dfrac{2x - 3}{x + 5}$
21. $\lim_{x\to 7} \dfrac{3x}{\sqrt{x + 2}}$
22. $\lim_{x\to 2} \dfrac{\sqrt{x + 2}}{x - 4}$

In Exercises 23–26, find the limits.

23. $f(x) = 5 - x,\ g(x) = x^3$
   (a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 4} g(x)$ (c) $\lim_{x\to 1} g(f(x))$

24. $f(x) = x + 7,\ g(x) = x^2$
   (a) $\lim_{x\to -3} f(x)$ (b) $\lim_{x\to 4} g(x)$ (c) $\lim_{x\to -3} g(f(x))$

25. $f(x) = 4 - x^2,\ g(x) = \sqrt{x + 1}$
   (a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 3} g(x)$ (c) $\lim_{x\to 1} g(f(x))$

26. $f(x) = 2x^2 - 3x + 1,\ g(x) = \sqrt[3]{x + 6}$
   (a) $\lim_{x\to 4} f(x)$ (b) $\lim_{x\to 21} g(x)$ (c) $\lim_{x\to 4} g(f(x))$

In Exercises 27–36, find the limit of the trigonometric function.

27. $\lim_{x\to \pi/2} \sin x$
28. $\lim_{x\to \pi} \tan x$
29. $\lim_{x\to 1} \cos \dfrac{\pi x}{3}$
30. $\lim_{x\to 2} \sin \dfrac{\pi x}{2}$
31. $\lim_{x\to 0} \sec 2x$
32. $\lim_{x\to \pi} \cos 3x$
33. $\lim_{x\to 5\pi/6} \sin x$
34. $\lim_{x\to 5\pi/3} \cos x$
35. $\lim_{x\to 3} \tan\left(\dfrac{\pi x}{4}\right)$
36. $\lim_{x\to 7} \sec\left(\dfrac{\pi x}{6}\right)$

In Exercises 37–40, use the information to evaluate the limits.

37. $\lim_{x\to c} f(x) = 3$
   $\lim_{x\to c} g(x) = 2$
   (a) $\lim_{x\to c} [5g(x)]$
   (b) $\lim_{x\to c} [f(x) + g(x)]$
   (c) $\lim_{x\to c} [f(x)g(x)]$
   (d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

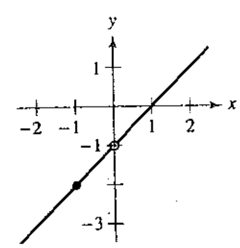
38. $\lim_{x\to c} f(x) = \frac{3}{2}$
   $\lim_{x\to c} g(x) = \frac{1}{2}$
   (a) $\lim_{x\to c} [4f(x)]$
   (b) $\lim_{x\to c} [f(x) + g(x)]$
   (c) $\lim_{x\to c} [f(x)g(x)]$
   (d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

39. $\lim_{x\to c} f(x) = 4$
   (a) $\lim_{x\to c} [f(x)]^3$
   (b) $\lim_{x\to c} \sqrt{f(x)}$
   (c) $\lim_{x\to c} [3f(x)]$
   (d) $\lim_{x\to c} [f(x)]^{3/2}$

40. $\lim_{x\to c} f(x) = 27$
   (a) $\lim_{x\to c} \sqrt[3]{f(x)}$
   (b) $\lim_{x\to c} \dfrac{f(x)}{18}$
   (c) $\lim_{x\to c} [f(x)]^2$
   (d) $\lim_{x\to c} [f(x)]^{2/3}$

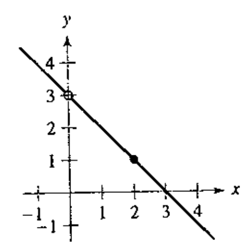
In Exercises 41–44, use the graph to determine the limit visually (if it exists). Write a simpler function that agrees with the given function at all but one point.

41. $g(x) = \dfrac{x^2 - x}{x}$



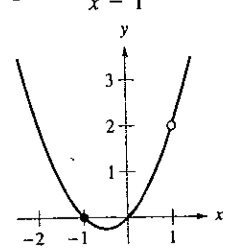
   (a) $\lim_{x\to 0} g(x)$
   (b) $\lim_{x\to -1} g(x)$

42. $h(x) = \dfrac{-x^2 + 3x}{x}$



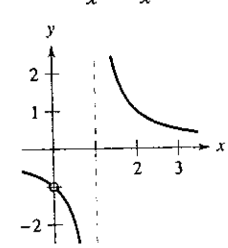
   (a) $\lim_{x\to 2} h(x)$
   (b) $\lim_{x\to 0} h(x)$

43. $g(x) = \dfrac{x^3 - x}{x - 1}$



   (a) $\lim_{x\to 1} g(x)$
   (b) $\lim_{x\to -1} g(x)$

44. $f(x) = \dfrac{x}{x^2 - x}$



   (a) $\lim_{x\to 1} f(x)$
   (b) $\lim_{x\to 0} f(x)$

In Exercises 45–48, find the limit of the function (if it exists). Write a simpler function that agrees with the given function at all but one point. Use a graphing utility to confirm your result.

45. $\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$

46. $\lim_{x \to -1} \frac{2x^2 - x - 3}{x + 1}$

47. $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$

48. $\lim_{x \to -1} \frac{x^3 + 1}{x + 1}$

In Exercises 49–64, find the limit (if it exists).

49. $\lim_{x \to 0} \frac{x}{x^2 - x}$

50. $\lim_{x \to 0} \frac{3x}{x^2 + 2x}$

51. $\lim_{x \to 4} \frac{x - 4}{x^2 - 16}$

52. $\lim_{x \to 3} \frac{3 - x}{x^2 - 9}$

53. $\lim_{x \to -3} \frac{x^2 + x - 6}{x^2 - 9}$

54. $\lim_{x \to 4} \frac{x^2 - 5x + 4}{x^2 - 2x - 8}$

55. $\lim_{x \to 4} \frac{\sqrt{x + 5} - 3}{x - 4}$

56. $\lim_{x \to 3} \frac{\sqrt{x + 1} - 2}{x - 3}$

57. $\lim_{x \to 0} \frac{\sqrt{x + 5} - \sqrt{5}}{x}$

58. $\lim_{x \to 0} \frac{\sqrt{2 + x} - \sqrt{2}}{x}$

59. $\lim_{x \to 0} \frac{[1/(3 + x)] - (1/3)}{x}$

60. $\lim_{x \to 0} \frac{[1/(x + 4)] - (1/4)}{x}$

61. $\lim_{\Delta x \to 0} \frac{2(x + \Delta x) - 2x}{\Delta x}$

62. $\lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$

63. $\lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x}$

64. $\lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$

In Exercises 65–76, determine the limit of the trigonometric function (if it exists).

65. $\lim_{x \to 0} \frac{\sin x}{5x}$

66. $\lim_{x \to 0} \frac{3(1 - \cos x)}{x}$

67. $\lim_{x \to 0} \frac{\sin x(1 - \cos x)}{x^2}$

68. $\lim_{\theta \to 0} \frac{\cos \theta \tan \theta}{\theta}$

69. $\lim_{x \to 0} \frac{\sin^2 x}{x}$

70. $\lim_{x \to 0} \frac{\tan^2 x}{x}$

71. $\lim_{h \to 0} \frac{(1 - \cos h)^2}{h}$

72. $\lim_{\phi \to \pi} \phi \sec \phi$

73. $\lim_{x \to \pi/2} \frac{\cos x}{\cot x}$

74. $\lim_{x \to \pi/4} \frac{1 - \tan x}{\sin x - \cos x}$

75. $\lim_{t \to 0} \frac{\sin 3t}{2t}$

76. $\lim_{x \to 0} \frac{\sin 2x}{\sin 3x}$ $\left[\textit{Hint: } \text{Find } \lim_{x \to 0} \left(\frac{2 \sin 2x}{2x}\right)\left(\frac{3x}{3 \sin 3x}\right).\right]$

Graphical, Numerical, and Analytic Analysis **In Exercises 77–84, use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.**

77. $\lim_{x \to 0} \frac{\sqrt{x + 2} - \sqrt{2}}{x}$

78. $\lim_{x \to 16} \frac{4 - \sqrt{x}}{x - 16}$

79. $\lim_{x \to 0} \frac{[1/(2 + x)] - (1/2)}{x}$

80. $\lim_{x \to 2} \frac{x^5 - 32}{x - 2}$

81. $\lim_{t \to 0} \frac{\sin 3t}{t}$

82. $\lim_{x \to 0} \frac{\cos x - 1}{2x^2}$

83. $\lim_{x \to 0} \frac{\sin x^2}{x}$

84. $\lim_{x \to 0} \frac{\sin x}{\sqrt[3]{x}}$

In Exercises 85–88, find $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$.

85. $f(x) = 3x - 2$

86. $f(x) = \sqrt{x}$

87. $f(x) = \frac{1}{x + 3}$

88. $f(x) = x^2 - 4x$

In Exercises 89 and 90, use the Squeeze Theorem to fin $\lim_{x \to c} f(x)$.

89. $c = 0$

$4 - x^2 \le f(x) \le 4 + x^2$

90. $c = a$

$b - |x - a| \le f(x) \le b + |x - a|$

In Exercises 91–96, use a graphing utility to graph the give function and the equations $y = |x|$ and $y = -|x|$ in the sam viewing window. Using the graphs to observe the Squeez Theorem visually, find $\lim_{x \to 0} f(x)$.

91. $f(x) = x \cos x$

92. $f(x) = |x \sin x|$

93. $f(x) = |x| \sin x$

94. $f(x) = |x| \cos x$

95. $f(x) = x \sin \frac{1}{x}$

96. $h(x) = x \cos \frac{1}{x}$

WRITING ABOUT CONCEPTS

97. In the context of finding limits, discuss what is meant by two functions that agree at all but one point.
98. Give an example of two functions that agree at all but one point.
99. What is meant by an indeterminate form?
100. In your own words, explain the Squeeze Theorem.

101. *Writing* Use a graphing utility to graph

$$f(x) = x, \quad g(x) = \sin x, \quad \text{and} \quad h(x) = \frac{\sin x}{x}$$

in the same viewing window. Compare the magnitudes of $f($ and $g(x)$ when x is close to 0. Use the comparison to write short paragraph explaining why

$$\lim_{x \to 0} h(x) = 1.$$

102. *Writing* Use a graphing utility to graph

$$f(x) = x, \ g(x) = \sin^2 x, \ \text{and } h(x) = \frac{\sin^2 x}{x}$$

in the same viewing window. Compare the magnitudes of $f(x)$ and $g(x)$ when x is close to 0. Use the comparison to write a short paragraph explaining why $\lim_{x\to 0} h(x) = 0$.

Free-Falling Object **In Exercises 103 and 104, use the position function $s(t) = -16t^2 + 500$, which gives the height (in feet) of an object that has fallen for t seconds from a height of 500 feet. The velocity at time $t = a$ seconds is given by**

$$\lim_{t\to a} \frac{s(a) - s(t)}{a - t}.$$

103. If a construction worker drops a wrench from a height of 500 feet, how fast will the wrench be falling after 2 seconds?

104. If a construction worker drops a wrench from a height of 500 feet, when will the wrench hit the ground? At what velocity will the wrench impact the ground?

Free-Falling Object **In Exercises 105 and 106, use the position function $s(t) = -4.9t^2 + 200$, which gives the height (in meters) of an object that has fallen from a height of 200 meters. The velocity at time $t = a$ seconds is given by**

$$\lim_{t\to a} \frac{s(a) - s(t)}{a - t}.$$

105. Find the velocity of the object when $t = 3$.

106. At what velocity will the object impact the ground?

107. Find two functions f and g such that $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist, but $\lim_{x\to 0} [f(x) + g(x)]$ does exist.

108. Prove that if $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} [f(x) + g(x)]$ does not exist, then $\lim_{x\to c} g(x)$ does not exist.

109. Prove Property 1 of Theorem 1.1.

110. Prove Property 3 of Theorem 1.1. (You may use Property 3 of Theorem 1.2.)

111. Prove Property 1 of Theorem 1.2.

112. Prove that if $\lim_{x\to c} f(x) = 0$, then $\lim_{x\to c} |f(x)| = 0$.

113. Prove that if $\lim_{x\to c} f(x) = 0$ and $|g(x)| \le M$ for a fixed number M and all $x \ne c$, then $\lim_{x\to c} f(x)g(x) = 0$.

114. (a) Prove that if $\lim_{x\to c} |f(x)| = 0$, then $\lim_{x\to c} f(x) = 0$.
(*Note:* This is the converse of Exercise 112.)
(b) Prove that if $\lim_{x\to c} f(x) = L$, then $\lim_{x\to c} |f(x)| = |L|$.
[*Hint:* Use the inequality $||f(x)| - |L|| \le |f(x) - L|$.]

115. *Think About It* Find a function f to show that the converse of Exercise 114(b) is not true. [*Hint:* Find a function f such that $\lim_{x\to c} |f(x)| = |L|$ but $\lim_{x\to c} f(x)$ does not exist.]

CAPSTONE

116. Let $f(x) = \begin{cases} 3, & x \ne 2 \\ 5, & x = 2 \end{cases}$. Find $\lim_{x\to 2} f(x)$.

True or False? **In Exercises 117–122, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

117. $\lim_{x\to 0} \frac{|x|}{x} = 1$

118. $\lim_{x\to \pi} \frac{\sin x}{x} = 1$

119. If $f(x) = g(x)$ for all real numbers other than $x = 0$, and $\lim_{x\to 0} f(x) = L$, then $\lim_{x\to 0} g(x) = L$.

120. If $\lim_{x\to c} f(x) = L$, then $f(c) = L$.

121. $\lim_{x\to 2} f(x) = 3$, where $f(x) = \begin{cases} 3, & x \le 2 \\ 0, & x > 2 \end{cases}$

122. If $f(x) < g(x)$ for all $x \ne a$, then $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$.

123. Prove the second part of Theorem 1.9.

$$\lim_{x\to 0} \frac{1 - \cos x}{x} = 0$$

124. Let $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

and

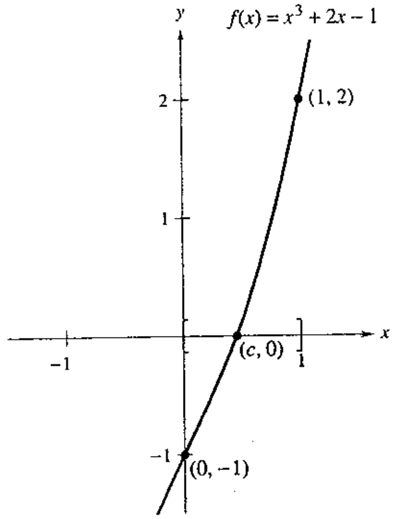
$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational.} \end{cases}$$

Find (if possible) $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$.

125. *Graphical Reasoning* Consider $f(x) = \dfrac{\sec x - 1}{x^2}$.
(a) Find the domain of f.
(b) Use a graphing utility to graph f. Is the domain of f obvious from the graph? If not, explain.
(c) Use the graph of f to approximate $\lim_{x\to 0} f(x)$.
(d) Confirm your answer to part (c) analytically.

126. *Approximation*
(a) Find $\lim_{x\to 0} \dfrac{1 - \cos x}{x^2}$.
(b) Use your answer to part (a) to derive the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ for x near 0.
(c) Use your answer to part (b) to approximate $\cos(0.1)$.
(d) Use a calculator to approximate $\cos(0.1)$ to four decimal places. Compare the result with part (c).

127. *Think About It* When using a graphing utility to generate a table to approximate $\lim_{x\to 0}[(\sin x)/x]$, a student concluded that the limit was 0.01745 rather than 1. Determine the probable cause of the error.



f is continuous on $[0, 1]$ with $f(0) < 0$ and $f(1) > 0$.
Figure 1.37

EXAMPLE 8 An Application of the Intermediate Value Theorem

Use the Intermediate Value Theorem to show that the polynomial functic $f(x) = x^3 + 2x - 1$ has a zero in the interval $[0, 1]$.

Solution Note that f is continuous on the closed interval $[0, 1]$. Because

$$f(0) = 0^3 + 2(0) - 1 = -1 \quad \text{and} \quad f(1) = 1^3 + 2(1) - 1 = 2$$

it follows that $f(0) < 0$ and $f(1) > 0$. You can therefore apply the Intermediate Valı Theorem to conclude that there must be some c in $[0, 1]$ such that

$$f(c) = 0 \qquad f \text{ has a zero in the closed interval } [0, 1].$$

as shown in Figure 1.37.

The **bisection method** for approximating the real zeros of a continuous functic is similar to the method used in Example 8. If you know that a zero exists in the closi interval $[a, b]$, the zero must lie in the interval $[a, (a + b)/2]$ or $[(a + b)/2, b]$. Fro the sign of $f([a + b]/2)$, you can determine which interval contains the zero. I repeatedly bisecting the interval, you can "close in" on the zero of the function.

TECHNOLOGY You can also use the *zoom* feature of a graphing utility to appro: mate the real zeros of a continuous function. By repeatedly zooming in on t point where the graph crosses the x-axis, and adjusting the x-axis scale, you c approximate the zero of the function to any desired accuracy. The zero $x^3 + 2x - 1$ is approximately 0.453, as shown in Figure 1.38.

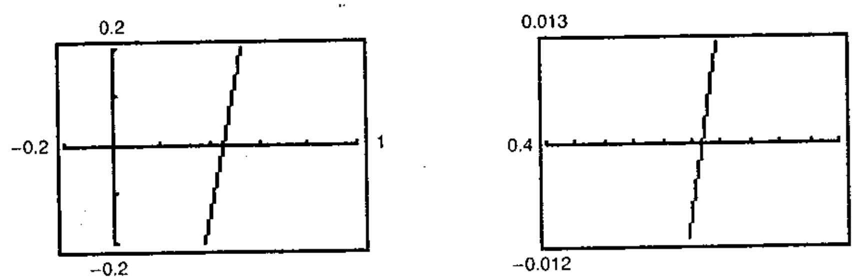


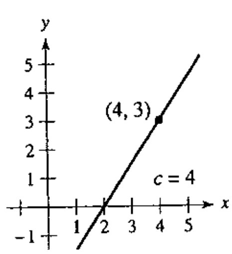
Figure 1.38 Zooming in on the zero of $f(x) = x^3 + 2x - 1$

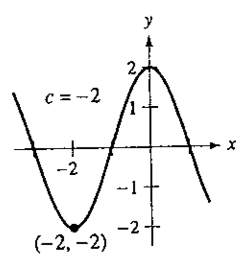
1.4 Exercises

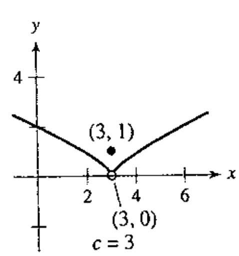
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, use the graph to determine the limit, and discuss the continuity of the function.

(a) $\lim_{x \to c^+} f(x)$ (b) $\lim_{x \to c^-} f(x)$ (c) $\lim_{x \to c} f(x)$

1.


2.


3.


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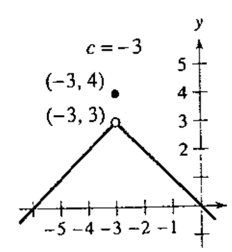
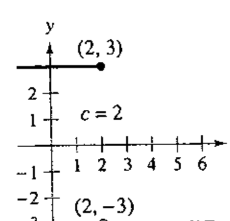
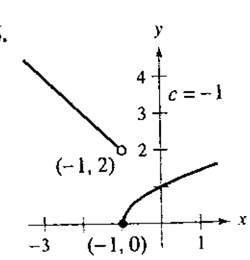


5.


6.


[Exe]rcises 7–26, find the limit (if it exists). If it does not exist, [expla]in why.

$\lim_{x\to 8^+} \frac{1}{x+8}$

8. $\lim_{x\to 5^-} \frac{-3}{x+5}$

$\lim_{x\to 5^+} \frac{x-5}{x^2-25}$

10. $\lim_{x\to 2^+} \frac{2-x}{x^2-4}$

$\lim_{x\to -3^-} \frac{x}{\sqrt{x^2-9}}$

12. $\lim_{x\to 9^-} \frac{\sqrt{x}-3}{x-9}$

$\lim_{x\to 0^-} \frac{|x|}{x}$

$\lim_{x\to 10^+} \frac{|x-10|}{x-10}$

$\lim_{\Delta x\to 0^-} \dfrac{\frac{1}{x+\Delta x}-\frac{1}{x}}{\Delta x}$

$\lim_{\Delta x\to 0^+} \frac{(x+\Delta x)^2+x+\Delta x-(x^2+x)}{\Delta x}$

$\lim_{x\to 3} f(x)$, where $f(x)=\begin{cases}\frac{x+2}{2}, & x\le 3\\ \frac{12-2x}{3}, & x>3\end{cases}$

$\lim_{x\to 2} f(x)$, where $f(x)=\begin{cases}x^2-4x+6, & x<2\\ -x^2+4x-2, & x\ge 2\end{cases}$

$\lim_{x\to 1} f(x)$, where $f(x)=\begin{cases}x^3+1, & x<1\\ x+1, & x\ge 1\end{cases}$

$\lim_{x\to 1^+} f(x)$, where $f(x)=\begin{cases}x, & x\le 1\\ 1-x, & x>1\end{cases}$

$\lim \cot x$

$\lim_{x\to \pi/2} \sec x$

$\lim (5[\![x]\!]-7)$

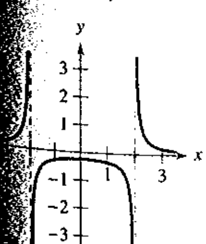
$\lim (2x-[\![x]\!])$

$\lim (2-[\![-x]\!])$

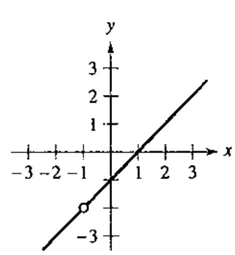
$\lim \left(1-\left[\!\!\left[-\frac{x}{2}\right]\!\!\right]\right)$

[Exe]rcises 27–30, discuss the continuity of each function.

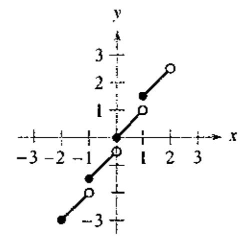
$f(x)=\frac{1}{x^2-4}$



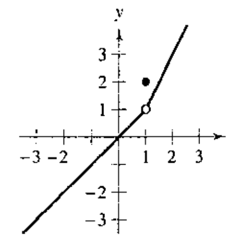
28. $f(x)=\frac{x^2-1}{x+1}$



29. $f(x)=\frac{1}{2}[\![x]\!]+x$



30. $f(x)=\begin{cases}x, & x<1\\ 2, & x=1\\ 2x-1, & x>1\end{cases}$



In Exercises 31–34, discuss the continuity of the function on the closed interval.

Function	*Interval*
31. $g(x)=\sqrt{49-x^2}$	$[-7,7]$
32. $f(t)=3-\sqrt{9-t^2}$	$[-3,3]$
33. $f(x)=\begin{cases}3-x, & x\le 0\\ 3+\frac{1}{2}x, & x>0\end{cases}$	$[-1,4]$
34. $g(x)=\frac{1}{x^2-4}$	$[-1,2]$

In Exercises 35–60, find the x-values (if any) at which f is not continuous. Which of the discontinuities are removable?

35. $f(x)=\frac{6}{x}$

36. $f(x)=\frac{3}{x-2}$

37. $f(x)=x^2-9$

38. $f(x)=x^2-2x+1$

39. $f(x)=\frac{1}{4-x^2}$

40. $f(x)=\frac{1}{x^2+1}$

41. $f(x)=3x-\cos x$

42. $f(x)=\cos\frac{\pi x}{2}$

43. $f(x)=\frac{x}{x^2-x}$

44. $f(x)=\frac{x}{x^2-1}$

45. $f(x)=\frac{x}{x^2+1}$

46. $f(x)=\frac{x-6}{x^2-36}$

47. $f(x)=\frac{x+2}{x^2-3x-10}$

48. $f(x)=\frac{x-1}{x^2+x-2}$

49. $f(x)=\frac{|x+7|}{x+7}$

50. $f(x)=\frac{|x-8|}{x-8}$

51. $f(x)=\begin{cases}x, & x\le 1\\ x^2, & x>1\end{cases}$

52. $f(x)=\begin{cases}-2x+3, & x<1\\ x^2, & x\ge 1\end{cases}$

53. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

54. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

55. $f(x) = \begin{cases} \tan \frac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

56. $f(x) = \begin{cases} \csc \frac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases}$

57. $f(x) = \csc 2x$

58. $f(x) = \tan \frac{\pi x}{2}$

59. $f(x) = [\![x - 8]\!]$

60. $f(x) = 5 - [\![x]\!]$

In Exercises 61 and 62, use a graphing utility to graph the function. From the graph, estimate

$$\lim_{x \to 0^+} f(x) \quad \text{and} \quad \lim_{x \to 0^-} f(x).$$

Is the function continuous on the entire real line? Explain.

61. $f(x) = \dfrac{|x^2 - 4|x}{x + 2}$

62. $f(x) = \dfrac{|x^2 + 4x|(x + 2)}{x + 4}$

In Exercises 63–68, find the constant a, or the constants a and b, such that the function is continuous on the entire real line.

63. $f(x) = \begin{cases} 3x^2, & x \ge 1 \\ ax - 4, & x < 1 \end{cases}$

64. $f(x) = \begin{cases} 3x^3, & x \le 1 \\ ax + 5, & x > 1 \end{cases}$

65. $f(x) = \begin{cases} x^3, & x \le 2 \\ ax^2, & x > 2 \end{cases}$

66. $g(x) = \begin{cases} \dfrac{4 \sin x}{x}, & x < 0 \\ a - 2x, & x \ge 0 \end{cases}$

67. $f(x) = \begin{cases} 2, & x \le -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$

68. $g(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a}, & x \ne a \\ 8, & x = a \end{cases}$

In Exercises 69–72, discuss the continuity of the composite function $h(x) = f(g(x))$.

69. $f(x) = x^2$
$g(x) = x - 1$

70. $f(x) = \dfrac{1}{\sqrt{x}}$
$g(x) = x - 1$

71. $f(x) = \dfrac{1}{x - 6}$
$g(x) = x^2 + 5$

72. $f(x) = \sin x$
$g(x) = x^2$

In Exercises 73–76, use a graphing utility to graph the function. Use the graph to determine any x-values at which the function is not continuous.

73. $f(x) = [\![x]\!] - x$

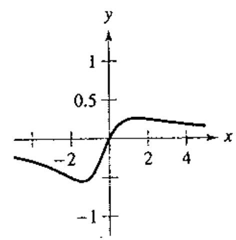
74. $h(x) = \dfrac{1}{x^2 - x - 2}$

75. $g(x) = \begin{cases} x^2 - 3x, & x > 4 \\ 2x - 5, & x \le 4 \end{cases}$

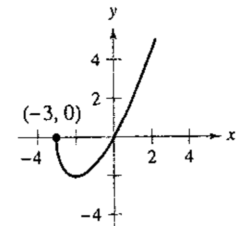
76. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$

In Exercises 77–80, describe the interval(s) on which the function is continuous.

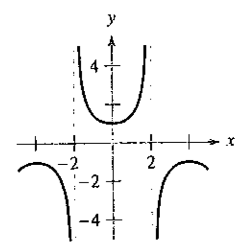
77. $f(x) = \dfrac{x}{x^2 + x + 2}$



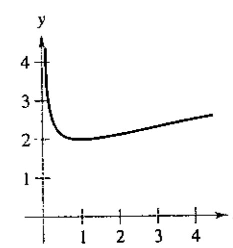
78. $f(x) = x\sqrt{x + 3}$



79. $f(x) = \sec \dfrac{\pi x}{4}$



80. $f(x) = \dfrac{x + 1}{\sqrt{x}}$



Writing **In Exercises 81 and 82, use a graphing utility to graph the function on the interval $[-4, 4]$. Does the graph of the function appear to be continuous on this interval? Is the function continuous on $[-4, 4]$? Write a short paragraph about the importance of examining a function analytically as well as graphically.**

81. $f(x) = \dfrac{\sin x}{x}$

82. $f(x) = \dfrac{x^3 - 8}{x - 2}$

Writing **In Exercises 83–86, explain why the function has a zero in the given interval.**

Function	*Interval*
83. $f(x) = \frac{1}{12}x^4 - x^3 + 4$	$[1, 2]$
84. $f(x) = x^3 + 5x - 3$	$[0, 1]$
85. $f(x) = x^2 - 2 - \cos x$	$[0, \pi]$
86. $f(x) = -\dfrac{5}{x} + \tan\left(\dfrac{\pi x}{10}\right)$	$[1, 4]$

In Exercises 87–90, use the Intermediate Value Theorem and a graphing utility to approximate the zero of the function in the interval [0, 1]. Repeatedly "zoom in" on the graph of the function to approximate the zero accurate to two decimal places. Use the *zero* or *root* feature of the graphing utility to approximate the zero accurate to four decimal places.

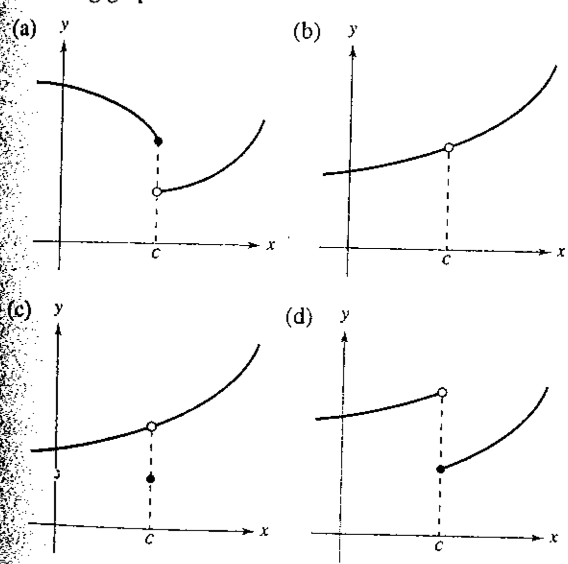
87. $f(x) = x^3 + x - 1$
88. $f(x) = x^3 + 5x - 3$
89. $g(t) = 2\cos t - 3t$
90. $h(\theta) = 1 + \theta - 3\tan\theta$

In Exercises 91–94, verify that the Intermediate Value Theorem applies to the indicated interval and find the value of c guaranteed by the theorem.

91. $f(x) = x^2 + x - 1, \quad [0, 5], \quad f(c) = 11$
92. $f(x) = x^2 - 6x + 8, \quad [0, 3], \quad f(c) = 0$
93. $f(x) = x^3 - x^2 + x - 2, \quad [0, 3], \quad f(c) = 4$
94. $f(x) = \dfrac{x^2 + x}{x - 1}, \quad \left[\dfrac{5}{2}, 4\right], \quad f(c) = 6$

WRITING ABOUT CONCEPTS

95. State how continuity is destroyed at $x = c$ for each of the following graphs.



96. Sketch the graph of any function f such that

$$\lim_{x\to 3^+} f(x) = 1 \quad \text{and} \quad \lim_{x\to 3^-} f(x) = 0.$$

Is the function continuous at $x = 3$? Explain.

97. If the functions f and g are continuous for all real x, is $f + g$ always continuous for all real x? Is f/g always continuous for all real x? If either is not continuous, give an example to verify your conclusion.

CAPSTONE

98. Describe the difference between a discontinuity that is removable and one that is nonremovable. In your explanation, give examples of the following descriptions.

(a) A function with a nonremovable discontinuity at $x = 4$

(b) A function with a removable discontinuity at $x = -4$

(c) A function that has both of the characteristics described in parts (a) and (b)

True or False? **In Exercises 99–102, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

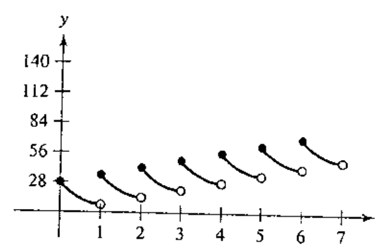
99. If $\lim_{x\to c} f(x) = L$ and $f(c) = L$, then f is continuous at c.

100. If $f(x) = g(x)$ for $x \neq c$ and $f(c) \neq g(c)$, then either f or g is not continuous at c.

101. A rational function can have infinitely many x-values at which it is not continuous.

102. The function $f(x) = |x - 1|/(x - 1)$ is continuous on $(-\infty, \infty)$.

103. ***Swimming Pool*** Every day you dissolve 28 ounces of chlorine in a swimming pool. The graph shows the amount of chlorine $f(t)$ in the pool after t days.



Estimate and interpret $\lim_{t\to 4^-} f(t)$ and $\lim_{t\to 4^+} f(t)$.

104. ***Think About It*** Describe how the functions

$f(x) = 3 + [\![x]\!]$

and

$g(x) = 3 - [\![-x]\!]$

differ.

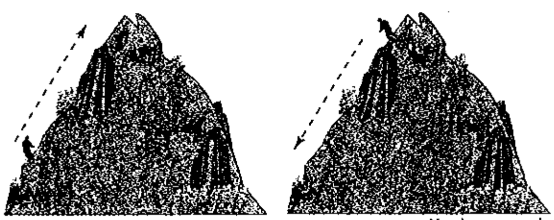
105. ***Telephone Charges*** A long distance phone service charges \$0.40 for the first 10 minutes and \$0.05 for each additional minute or fraction thereof. Use the greatest integer function to write the cost C of a call in terms of time t (in minutes). Sketch the graph of this function and discuss its continuity.

106. ***Inventory Management*** The number of units in inventory in a small company is given by

$$N(t) = 25\left(2\left[\!\left[\frac{t+2}{2}\right]\!\right] - t\right)$$

where t is the time in months. Sketch the graph of this function and discuss its continuity. How often must this company replenish its inventory?

107. ***Déjà Vu*** At 8:00 A.M. on Saturday a man begins running up the side of a mountain to his weekend campsite (see figure). On Sunday morning at 8:00 A.M. he runs back down the mountain. It takes him 20 minutes to run up, but only 10 minutes to run down. At some point on the way down, he realizes that he passed the same place at exactly the same time on Saturday. Prove that he is correct. [*Hint:* Let $s(t)$ and $r(t)$ be the position functions for the runs up and down, and apply the Intermediate Value Theorem to the function $f(t) = s(t) - r(t)$.]



Not drawn to scale

Saturday 8:00 A.M. Sunday 8:00 A.M.

108. ***Volume*** Use the Intermediate Value Theorem to show that for all spheres with radii in the interval $[5, 8]$, there is one with a volume of 1500 cubic centimeters.

109. Prove that if f is continuous and has no zeros on $[a, b]$, then either

$f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

110. Show that the Dirichlet function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

is not continuous at any real number.

111. Show that the function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ kx, & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at $x = 0$. (Assume that k is any nonzero real number.)

112. The **signum function** is defined by

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0. \end{cases}$$

Sketch a graph of $\operatorname{sgn}(x)$ and find the following (if possible).

(a) $\lim_{x\to 0^-} \operatorname{sgn}(x)$ (b) $\lim_{x\to 0^+} \operatorname{sgn}(x)$ (c) $\lim_{x\to 0} \operatorname{sgn}(x)$

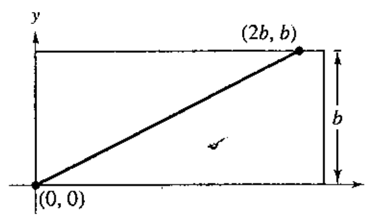
113. ***Modeling Data*** The table lists the speeds S (in feet pe second) of a falling object at various times t (in seconds).

0	5	10	15	20	25	30
0	48.2	53.5	55.2	55.9	56.2	56.3

(a) Create a line graph of the data.

(b) Does there appear to be a limiting speed of the object? there is a limiting speed, identify a possible cause.

114. ***Creating Models*** A swimmer crosses a pool of width b b swimming in a straight line from $(0, 0)$ to $(2b, b)$. (See figure



(a) Let f be a function defined as the y-coordinate of the poi on the long side of the pool that is nearest the swimmer any given time during the swimmer's crossing of the poc Determine the function f and sketch its graph. Is continuous? Explain.

(b) Let g be the minimum distance between the swimmer ar the long sides of the pool. Determine the function g ar sketch its graph. Is g continuous? Explain.

115. Find all values of c such that f is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} 1 - x^2, & x \le c \\ x, & x > c \end{cases}$$

116. Prove that for any real number y there exists x in $(-\pi/2, \pi/$ such that $\tan x = y$.

117. Let $f(x) = \left(\sqrt{x + c^2} - c\right)/x$, $c > 0$. What is the domain f? How can you define f at $x = 0$ in order for f to l continuous there?

118. Prove that if $\lim_{\Delta x\to 0} f(c + \Delta x) = f(c)$, then f is continuous at

119. Discuss the continuity of the function $h(x) = x[\![x]\!]$.

120. (a) Let $f_1(x)$ and $f_2(x)$ be continuous on the closed interv $[a, b]$. If $f_1(a) < f_2(a)$ and $f_1(b) > f_2(b)$, prove that the exists c between a and b such that $f_1(c) = f_2(c)$.

(b) Show that there exists c in $\left[0, \frac{\pi}{2}\right]$ such that $\cos x = x$. U a graphing utility to approximate c to three decimal place

PUTNAM EXAM CHALLENGE

121. Prove or disprove: if x and y are real numbers with $y \ge$ C and $y(y + 1) \le (x + 1)^2$, then $y(y - 1) \le x^2$.

122. Determine all polynomials $P(x)$ such that

$P(x^2 + 1) = (P(x))^2 + 1$ and $P(0) = 0$.

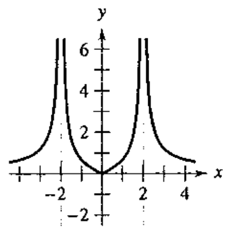
1.5 Exercises

In Exercises 1–4, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches 4 from the left and from the right.

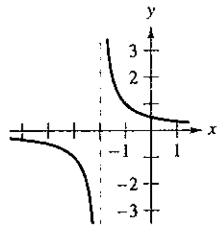
1. $f(x) = \dfrac{1}{x-4}$
2. $f(x) = \dfrac{-1}{x-4}$
3. $f(x) = \dfrac{1}{(x-4)^2}$
4. $f(x) = \dfrac{-1}{(x-4)^2}$

In Exercises 5–8, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -2 from the left and from the right.

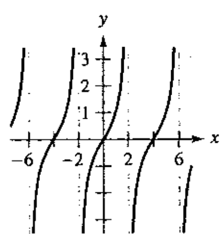
5. $f(x) = 2\left|\dfrac{x}{x^2-4}\right|$



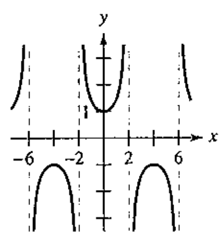
6. $f(x) = \dfrac{1}{x+2}$



7. $f(x) = \tan\dfrac{\pi x}{4}$



8. $f(x) = \sec\dfrac{\pi x}{4}$



Numerical and Graphical Analysis **In Exercises 9–12, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -3 from the left and from the right by completing the table. Use a graphing utility to graph the function to confirm your answer.**

x	-3.5	-3.1	-3.01	-3.001
$f(x)$				

x	-2.999	-2.99	-2.9	-2.5
$f(x)$				

9. $f(x) = \dfrac{1}{x^2-9}$
10. $f(x) = \dfrac{x}{x^2-9}$
11. $f(x) = \dfrac{x^2}{x^2-9}$
12. $f(x) = \sec\dfrac{\pi x}{6}$

In Exercises 13–32, find the vertical asymptotes (if any) of the graph of the function.

13. $f(x) = \dfrac{1}{x^2}$
14. $f(x) = \dfrac{4}{(x-2)^3}$
15. $f(x) = \dfrac{x^2}{x^2-4}$
16. $f(x) = \dfrac{-4x}{x^2+4}$
17. $g(t) = \dfrac{t-1}{t^2+1}$
18. $h(s) = \dfrac{2s-3}{s^2-25}$
19. $h(x) = \dfrac{x^2-2}{x^2-x-2}$
20. $g(x) = \dfrac{2+x}{x^2(1-x)}$
21. $T(t) = 1 - \dfrac{4}{t^2}$
22. $g(x) = \dfrac{\frac{1}{2}x^3 - x^2 - 4x}{3x^2-6x-24}$
23. $f(x) = \dfrac{3}{x^2+x-2}$
24. $f(x) = \dfrac{4x^2+4x-24}{x^4-2x^3-9x^2+18x}$
25. $g(x) = \dfrac{x^3+1}{x+1}$
26. $h(x) = \dfrac{x^2-4}{x^3+2x^2+x+2}$
27. $f(x) = \dfrac{x^2-2x-15}{x^3-5x^2+x-5}$
28. $h(t) = \dfrac{t^2-2t}{t^4-16}$
29. $f(x) = \tan \pi x$
30. $f(x) = \sec \pi x$
31. $s(t) = \dfrac{t}{\sin t}$
32. $g(\theta) = \dfrac{\tan\theta}{\theta}$

In Exercises 33–36, determine whether the graph of the function has a vertical asymptote or a removable discontinuity at $x = -1$. Graph the function using a graphing utility to confirm your answer.

33. $f(x) = \dfrac{x^2-1}{x+1}$
34. $f(x) = \dfrac{x^2-6x-7}{x+1}$
35. $f(x) = \dfrac{x^2+1}{x+1}$
36. $f(x) = \dfrac{\sin(x+1)}{x+1}$

In Exercises 37–54, find the limit (if it exists).

37. $\displaystyle\lim_{x\to -1^+} \frac{1}{x+1}$
38. $\displaystyle\lim_{x\to 1^-} \frac{-1}{(x-1)^2}$
39. $\displaystyle\lim_{x\to 2^+} \frac{x}{x-2}$
40. $\displaystyle\lim_{x\to 1^+} \frac{2+x}{1-x}$
41. $\displaystyle\lim_{x\to 1^+} \frac{x^2}{(x-1)^2}$
42. $\displaystyle\lim_{x\to 4^-} \frac{x^2}{x^2+16}$
43. $\displaystyle\lim_{x\to -3^-} \frac{x+3}{x^2+x-6}$
44. $\displaystyle\lim_{x\to (-1/2)^+} \frac{6x^2+x-1}{4x^2-4x-3}$
45. $\displaystyle\lim_{x\to 1} \frac{x-1}{(x^2+1)(x-1)}$
46. $\displaystyle\lim_{x\to 3} \frac{x-2}{x^2}$
47. $\displaystyle\lim_{x\to 0^-} \left(1+\frac{1}{x}\right)$
48. $\displaystyle\lim_{x\to 0^-} \left(x^2-\frac{1}{x}\right)$
49. $\displaystyle\lim_{x\to 0^+} \frac{2}{\sin x}$
50. $\displaystyle\lim_{x\to (\pi/2)^+} \frac{-2}{\cos x}$
51. $\displaystyle\lim_{x\to \pi} \frac{\sqrt{x}}{\csc x}$
52. $\displaystyle\lim_{x\to 0} \frac{x+2}{\cot x}$
53. $\displaystyle\lim_{x\to 1/2} x \sec \pi x$
54. $\displaystyle\lim_{x\to 1/2} x^2 \tan \pi x$

In Exercises 55–58, use a graphing utility to graph the function and determine the one-sided limit.

55. $f(x) = \dfrac{x^2 + x + 1}{x^3 - 1}$

$\lim_{x \to 1^-} f(x)$

56. $f(x) = \dfrac{x^3 - 1}{x^2 + x + 1}$

$\lim_{x \to 1^-} f(x)$

57. $f(x) = \dfrac{1}{x^2 - 25}$

$\lim_{x \to 5^-} f(x)$

58. $f(x) = \sec \dfrac{\pi x}{8}$

$\lim_{x \to 4^+} f(x)$

WRITING ABOUT CONCEPTS

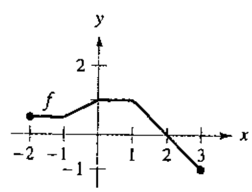
59. In your own words, describe the meaning of an infinite limit. Is ∞ a real number?

60. In your own words, describe what is meant by an asymptote of a graph.

61. Write a rational function with vertical asymptotes at $x = 6$ and $x = -2$, and with a zero at $x = 3$.

62. Does the graph of every rational function have a vertical asymptote? Explain.

63. Use the graph of the function f (see figure) to sketch the graph of $g(x) = 1/f(x)$ on the interval $[-2, 3]$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



CAPSTONE

64. Given a polynomial $p(x)$, is it true that the graph of the function given by $f(x) = \dfrac{p(x)}{x - 1}$ has a vertical asymptote at $x = 1$? Why or why not?

65. ***Relativity*** According to the theory of relativity, the mass m of a particle depends on its velocity v. That is,

$$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}$$

where m_0 is the mass when the particle is at rest and c is the speed of light. Find the limit of the mass as v approaches c^-.

66. ***Boyle's Law*** For a quantity of gas at a constant temperature, the pressure P is inversely proportional to the volume V. Find the limit of P as $V \to 0^+$.

67. ***Rate of Change*** A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of $\frac{1}{2}$ revolution per second. The rate at which the light beam moves along the wall is $r = 50\pi \sec^2 \theta$ ft/sec.

(a) Find the rate r when θ is $\pi/6$.

(b) Find the rate r when θ is $\pi/3$.

(c) Find the limit of r as $\theta \to (\pi/2)^-$.

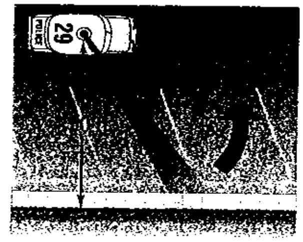


Figure for 67

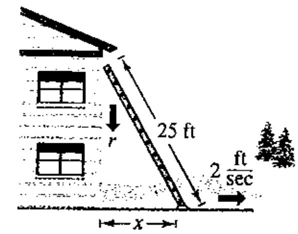


Figure for 68

68. ***Rate of Change*** A 25-foot ladder is leaning against a house (see figure). If the base of the ladder is pulled away from the house at a rate of 2 feet per second, the top will move down the wall at a rate of

$$r = \frac{2x}{\sqrt{625 - x^2}} \text{ ft/sec}$$

where x is the distance between the base of the ladder and the house.

(a) Find the rate r when x is 7 feet.

(b) Find the rate r when x is 15 feet.

(c) Find the limit of r as $x \to 25^-$.

69. ***Average Speed*** On a trip of d miles to another city, a truck driver's average speed was x miles per hour. On the return trip the average speed was y miles per hour. The average speed for the round trip was 50 miles per hour.

(a) Verify that $y = \dfrac{25x}{x - 25}$. What is the domain?

(b) Complete the table.

x	30	40	50	60
y				

Are the values of y different than you expected? Explain.

(c) Find the limit of y as $x \to 25^+$ and interpret its meaning.

70. ***Numerical and Graphical Analysis*** Use a graphing utility to complete the table for each function and graph each function to estimate the limit. What is the value of the limit when the power of x in the denominator is greater than 3?

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$							

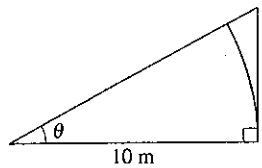
(a) $\lim_{x \to 0^+} \dfrac{x - \sin x}{x}$

(b) $\lim_{x \to 0^+} \dfrac{x - \sin x}{x^2}$

(c) $\lim_{x \to 0^+} \dfrac{x - \sin x}{x^3}$

(d) $\lim_{x \to 0^+} \dfrac{x - \sin x}{x^4}$

71. *Numerical and Graphical Analysis* Consider the shaded region outside the sector of a circle of radius 10 meters and inside a right triangle (see figure).



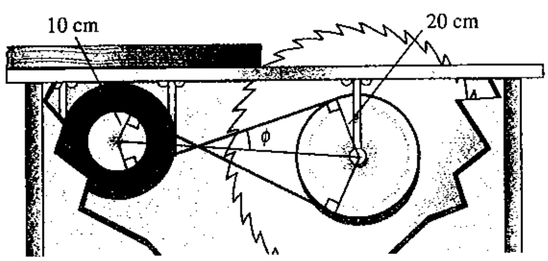
(a) Write the area $A = f(\theta)$ of the region as a function of θ. Determine the domain of the function.

(b) Use a graphing utility to complete the table and graph the function over the appropriate domain.

θ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$					

(c) Find the limit of A as $\theta \to (\pi/2)^-$.

72. *Numerical and Graphical Reasoning* A crossed belt connects a 20-centimeter pulley (10-cm radius) on an electric motor with a 40-centimeter pulley (20-cm radius) on a saw arbor (see figure). The electric motor runs at 1700 revolutions per minute.



(a) Determine the number of revolutions per minute of the saw.

(b) How does crossing the belt affect the saw in relation to the motor?

(c) Let L be the total length of the belt. Write L as a function of ϕ, where ϕ is measured in radians. What is the domain of the function? (*Hint:* Add the lengths of the straight sections of the belt and the length of the belt around each pulley.)

(d) Use a graphing utility to complete the table.

ϕ	0.3	0.6	0.9	1.2	1.5
L					

(e) Use a graphing utility to graph the function over the appro priate domain.

(f) Find $\lim_{\phi \to (\pi/2)^-} L$. Use a geometric argument as the basis c a second method of finding this limit.

(g) Find $\lim_{\phi \to 0^+} L$.

True or False? **In Exercises 73–76, determine whether tl statement is true or false. If it is false, explain why or give a example that shows it is false.**

73. The graph of a rational function has at least one vertic asymptote.

74. The graphs of polynomial functions have no vertic asymptotes.

75. The graphs of trigonometric functions have no vertic asymptotes.

76. If f has a vertical asymptote at $x = 0$, then f is undefined $x = 0$.

77. Find functions f and g such that $\lim_{x \to c} f(x) = \infty$ ai $\lim_{x \to c} g(x) = \infty$ but $\lim_{x \to c} [f(x) - g(x)] \neq 0$.

78. Prove the difference, product, and quotient properties Theorem 1.15.

79. Prove that if $\lim_{x \to c} f(x) = \infty$, then $\lim_{x \to c} \dfrac{1}{f(x)} = 0$.

80. Prove that if $\lim_{x \to c} \dfrac{1}{f(x)} = 0$, then $\lim_{x \to c} f(x)$ does not exist.

Infinite Limits **In Exercises 81 and 82, use the ε-δ definition infinite limits to prove the statement.**

81. $\lim_{x \to 3^+} \dfrac{1}{x - 3} = \infty$

82. $\lim_{x \to 5^-} \dfrac{1}{x - 5} = -\infty$

SECTION PROJECT

Graphs and Limits of Trigonometric Functions

Recall from Theorem 1.9 that the limit of $f(x) = (\sin x)/x$ as x approaches 0 is 1.

(a) Use a graphing utility to graph the function f on the interval $-\pi \leq x \leq \pi$. Explain how the graph helps confirm this theorem.

(b) Explain how you could use a table of values to confirm the value of this limit numerically.

(c) Graph $g(x) = \sin x$ by hand. Sketch a tangent line at the point $(0, 0)$ and visually estimate the slope of this tangent line.

(d) Let $(x, \sin x)$ be a point on the graph of g near $(0, 0)$, and wr a formula for the slope of the secant line joining $(x, \sin x)$ a $(0, 0)$. Evaluate this formula at $x = 0.1$ and $x = 0.01$. Then fi the exact slope of the tangent line to g at the point $(0, 0)$.

(e) Sketch the graph of the cosine function $h(x) = \cos x$. What the slope of the tangent line at the point $(0, 1)$? Use limits to fi this slope analytically.

(f) Find the slope of the tangent line to $k(x) = \tan x$ at $(0, 0)$.