2.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, assume that x and y are both differentiable functions of t and find the required values of dy/dt and dx/dt.

Equation	*Find*	*Given*
1. $y = \sqrt{x}$	(a) $\frac{dy}{dt}$ when $x = 4$	$\frac{dx}{dt} = 3$
	(b) $\frac{dx}{dt}$ when $x = 25$	$\frac{dy}{dt} = 2$
2. $y = 4(x^2 - 5x)$	(a) $\frac{dy}{dt}$ when $x = 3$	$\frac{dx}{dt} = 2$
	(b) $\frac{dx}{dt}$ when $x = 1$	$\frac{dy}{dt} = 5$
3. $xy = 4$	(a) $\frac{dy}{dt}$ when $x = 8$	$\frac{dx}{dt} = 10$
	(b) $\frac{dx}{dt}$ when $x = 1$	$\frac{dy}{dt} = -6$
4. $x^2 + y^2 = 25$	(a) $\frac{dy}{dt}$ when $x = 3, y = 4$	$\frac{dx}{dt} = 8$
	(b) $\frac{dx}{dt}$ when $x = 4, y = 3$	$\frac{dy}{dt} = -2$

In Exercises 5–8, a point is moving along the graph of the given function such that dx/dt is 2 centimeters per second. Find dy/dt for the given values of x.

5. $y = 2x^2 + 1$	(a) $x = -1$	(b) $x = 0$	(c) $x = 1$
6. $y = \dfrac{1}{1 + x^2}$	(a) $x = -2$	(b) $x = 0$	(c) $x = 2$
7. $y = \tan x$	(a) $x = -\frac{\pi}{3}$	(b) $x = -\frac{\pi}{4}$	(c) $x = 0$
8. $y = \cos x$	(a) $x = \frac{\pi}{6}$	(b) $x = \frac{\pi}{4}$	(c) $x = \frac{\pi}{3}$

WRITING ABOUT CONCEPTS

9. Consider the linear function $y = ax + b$. If x changes at a constant rate, does y change at a constant rate? If so, does it change at the same rate as x? Explain.

10. In your own words, state the guidelines for solving related-rate problems.

11. Find the rate of change of the distance between the origin and a moving point on the graph of $y = x^2 + 1$ if $dx/dt = 2$ centimeters per second.

12. Find the rate of change of the distance between the origin and a moving point on the graph of $y = \sin x$ if $dx/dt = 2$ centimeters per second.

13. ***Area*** The radius r of a circle is increasing at a rate of 4 centimeters per minute. Find the rates of change of the area when (a) $r = 8$ centimeters and (b) $r = 32$ centimeters.

14. ***Area*** Let A be the area of a circle of radius r that is changing with respect to time. If dr/dt is constant, is dA/dt constant? Explain.

15. ***Area*** The included angle of the two sides of constant equal length s of an isosceles triangle is θ.
    (a) Show that the area of the triangle is given by $A = \frac{1}{2}s^2 \sin \theta$.
    (b) If θ is increasing at the rate of $\frac{1}{2}$ radian per minute, find the rates of change of the area when $\theta = \pi/6$ and $\theta = \pi/3$.
    (c) Explain why the rate of change of the area of the triangle is not constant even though $d\theta/dt$ is constant.

16. ***Volume*** The radius r of a sphere is increasing at a rate of 3 inches per minute.
    (a) Find the rates of change of the volume when $r = 9$ inches and $r = 36$ inches.
    (b) Explain why the rate of change of the volume of the sphere is not constant even though dr/dt is constant.

17. ***Volume*** A spherical balloon is inflated with gas at the rate of 800 cubic centimeters per minute. How fast is the radius of the balloon increasing at the instant the radius is (a) 30 centimeters and (b) 60 centimeters?

18. ***Volume*** All edges of a cube are expanding at a rate of 6 centimeters per second. How fast is the volume changing when each edge is (a) 2 centimeters and (b) 10 centimeters?

19. ***Surface Area*** The conditions are the same as in Exercise 18. Determine how fast the *surface area* is changing when each edge is (a) 2 centimeters and (b) 10 centimeters.

20. ***Volume*** The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Find the rates of change of the volume if dr/dt is 2 inches per minute and $h = 3r$ when (a) $r = 6$ inches and (b) $r = 24$ inches.

21. ***Volume*** At a sand and gravel plant, sand is falling off a conveyor and onto a conical pile at a rate of 10 cubic feet per minute. The diameter of the base of the cone is approximately three times the altitude. At what rate is the height of the pile changing when the pile is 15 feet high?

22. ***Depth*** A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.

23. ***Depth*** A swimming pool is 12 meters long, 6 meters wide, 1 meter deep at the shallow end, and 3 meters deep at the deep end (see figure on next page). Water is being pumped into the pool at $\frac{1}{4}$ cubic meter per minute, and there is 1 meter of water at the deep end.
    (a) What percent of the pool is filled?
    (b) At what rate is the water level rising?

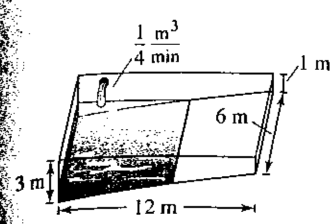


Figure for 23

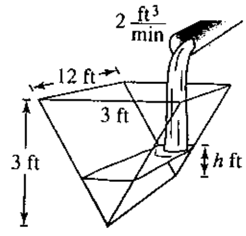


Figure for 24

Depth A trough is 12 feet long and 3 feet across the top (see figure). Its ends are isosceles triangles with altitudes of 3 feet.

(a) If water is being pumped into the trough at 2 cubic feet per minute, how fast is the water level rising when the depth h is 1 foot?

(b) If the water is rising at a rate of $\frac{3}{8}$ inch per minute when $h = 2$, determine the rate at which water is being pumped into the trough.

Moving Ladder A ladder 25 feet long is leaning against the wall of a house (see figure). The base of the ladder is pulled away from the wall at a rate of 2 feet per second.

(a) How fast is the top of the ladder moving down the wall when its base is 7 feet, 15 feet, and 24 feet from the wall?

(b) Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 feet from the wall.

(c) Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 7 feet from the wall.

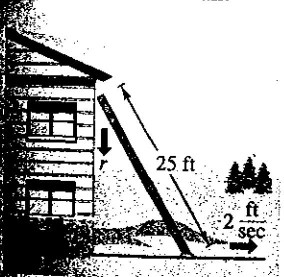


Figure for 25

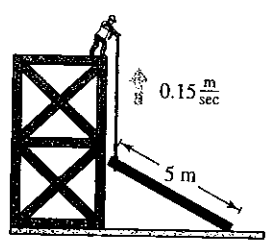


Figure for 26

...R FURTHER INFORMATION For more information on the ...matics of moving ladders, see the article "The Falling Ladder ...x" by Paul Scholten and Andrew Simoson in *The College ...matics Journal*. To view this article, go to the website ...athArticles.com.

...nstruction A construction worker pulls a five-meter plank ...the side of a building under construction by means of a rope ...d to one end of the plank (see figure). Assume the opposite ...d of the plank follows a path perpendicular to the wall of the ...ilding and the worker pulls the rope at a rate of 0.15 meter ...second. How fast is the end of the plank sliding along the ...und when it is 2.5 meters from the wall of the building?

27. *Construction* A winch at the top of a 12-meter building pulls a pipe of the same length to a vertical position, as shown in the figure. The winch pulls in rope at a rate of -0.2 meter per second. Find the rate of vertical change and the rate of horizontal change at the end of the pipe when $y = 6$.

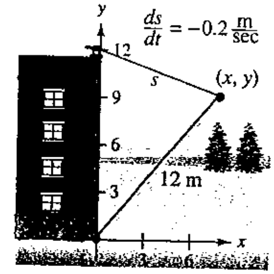


Figure for 27

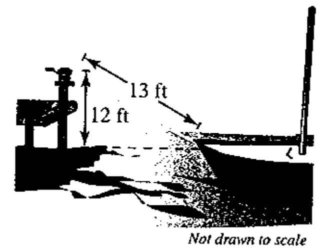


Figure for 28

28. *Boating* A boat is pulled into a dock by means of a winch 12 feet above the deck of the boat (see figure).

(a) The winch pulls in rope at a rate of 4 feet per second. Determine the speed of the boat when there is 13 feet of rope out. What happens to the speed of the boat as it gets closer to the dock?

(b) Suppose the boat is moving at a constant rate of 4 feet per second. Determine the speed at which the winch pulls in rope when there is a total of 13 feet of rope out. What happens to the speed at which the winch pulls in rope as the boat gets closer to the dock?

29. *Air Traffic Control* An air traffic controller spots two planes at the same altitude converging on a point as they fly at right angles to each other (see figure). One plane is 225 miles from the point moving at 450 miles per hour. The other plane is 300 miles from the point moving at 600 miles per hour.

(a) At what rate is the distance between the planes decreasing?

(b) How much time does the air traffic controller have to get one of the planes on a different flight path?

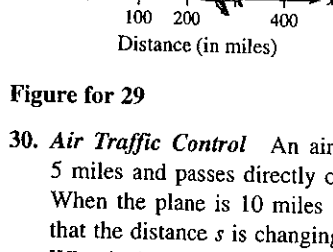


Figure for 29

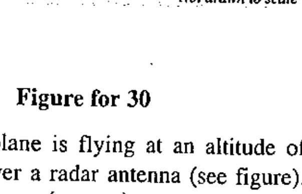


Figure for 30

30. *Air Traffic Control* An airplane is flying at an altitude of 5 miles and passes directly over a radar antenna (see figure). When the plane is 10 miles away ($s = 10$), the radar detects that the distance s is changing at a rate of 240 miles per hour. What is the speed of the plane?

31. ***Sports*** A baseball diamond has the shape of a square with sides 90 feet long (see figure). A player running from second base to third base at a speed of 25 feet per second is 20 feet from third base. At what rate is the player's distance s from home plate changing?

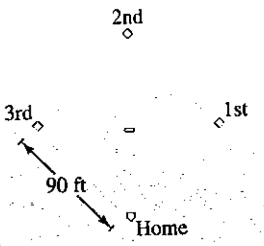


Figure for 31 and 32

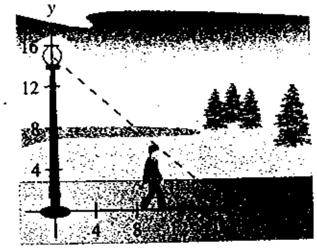


Figure for 33

32. ***Sports*** For the baseball diamond in Exercise 31, suppose the player is running from first to second at a speed of 25 feet per second. Find the rate at which the distance from home plate is changing when the player is 20 feet from second base.

33. ***Shadow Length*** A man 6 feet tall walks at a rate of 5 feet per second away from a light that is 15 feet above the ground (see figure). When he is 10 feet from the base of the light,

(a) at what rate is the tip of his shadow moving?

(b) at what rate is the length of his shadow changing?

34. ***Shadow Length*** Repeat Exercise 33 for a man 6 feet tall walking at a rate of 5 feet per second *toward* a light that is 20 feet above the ground (see figure).

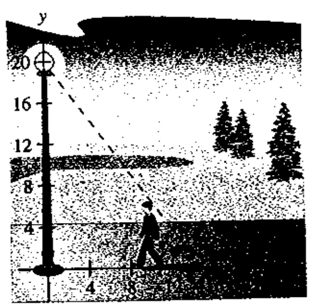


Figure for 34

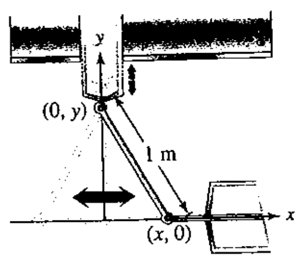


Figure for 35

35. ***Machine Design*** The endpoints of a movable rod of length 1 meter have coordinates $(x, 0)$ and $(0, y)$ (see figure). The position of the end on the x-axis is

$$x(t) = \frac{1}{2}\sin\frac{\pi t}{6}$$

where t is the time in seconds.

(a) Find the time of one complete cycle of the rod.

(b) What is the lowest point reached by the end of the rod on the y-axis?

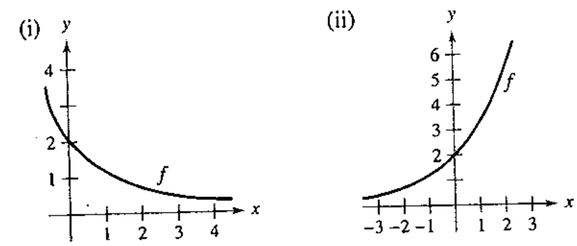
(c) Find the speed of the y-axis endpoint when the x-axis endpoint is $\left(\frac{1}{4}, 0\right)$.

36. ***Machine Design*** Repeat Exercise 35 for a position function of $x(t) = \frac{3}{5}\sin \pi t$. Use the point $\left(\frac{3}{10}, 0\right)$ for part (c).

37. ***Evaporation*** As a spherical raindrop falls, it reaches a layer of dry air and begins to evaporate at a rate that is proportional to its surface area $(S = 4\pi r^2)$. Show that the radius of the raindrop decreases at a constant rate.

CAPSTONE

38. Using the graph of f, (a) determine whether dy/dt is positive or negative given that dx/dt is negative, and (b) determine whether dx/dt is positive or negative given that dy/dt is positive.



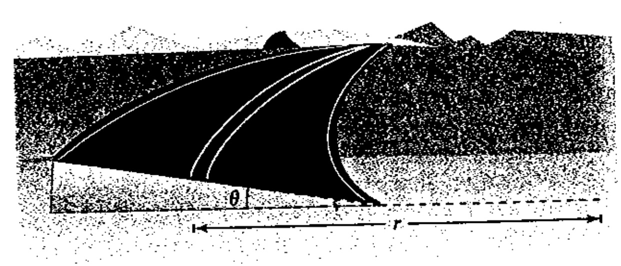
39. ***Electricity*** The combined electrical resistance R of R_1 and R_2, connected in parallel, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

where R, R_1, and R_2 are measured in ohms. R_1 and R_2 are increasing at rates of 1 and 1.5 ohms per second, respectively. At what rate is R changing when $R_1 = 50$ ohms and $R_2 = 75$ ohms?

40. ***Adiabatic Expansion*** When a certain polyatomic gas undergoes adiabatic expansion, its pressure p and volume V satisfy the equation $pV^{1.3} = k$, where k is a constant. Find the relationship between the related rates dp/dt and dV/dt.

41. ***Roadway Design*** Cars on a certain roadway travel on a circular arc of radius r. In order not to rely on friction alone to overcome the centrifugal force, the road is banked at an angle of magnitude θ from the horizontal (see figure). The banking angle must satisfy the equation $rg\tan\theta = v^2$, where v is the velocity of the cars and $g = 32$ feet per second per second is the acceleration due to gravity. Find the relationship between the related rates dv/dt and $d\theta/dt$.



42. ***Angle of Elevation*** A balloon rises at a rate of 4 meters per second from a point on the ground 50 meters from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 50 meters above the ground.

43. ***Angle of Elevation*** A fish is reeled in at a rate of 1 foot per second from a point 10 feet above the water (see figure). At what rate is the angle θ between the line and the water changing when there is a total of 25 feet of line from the end of the rod to the water?

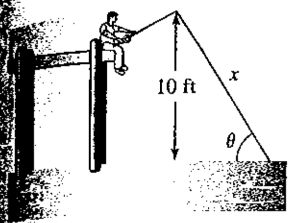


Figure for 43

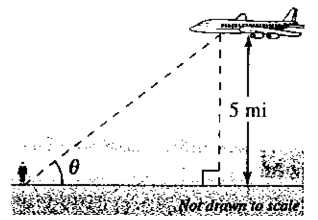


Figure for 44

44. ***Angle of Elevation*** An airplane flies at an altitude of 5 miles toward a point directly over an observer (see figure). The speed of the plane is 600 miles per hour. Find the rates at which the angle of elevation θ is changing when the angle is (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 75°$.

45. ***Linear vs. Angular Speed*** A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of 30 revolutions per minute. How fast is the light beam moving along the wall when the beam makes angles of (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 70°$ with the perpendicular line from the light to the wall?

Figure for 45

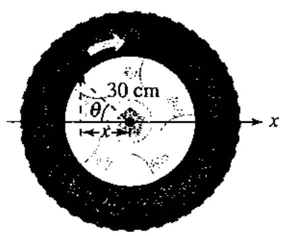


Figure for 46

46. ***Linear vs. Angular Speed*** A wheel of radius 30 centimeters revolves at a rate of 10 revolutions per second. A dot is painted at a point P on the rim of the wheel (see figure).

(a) Find dx/dt as a function of θ.

(b) Use a graphing utility to graph the function in part (a).

(c) When is the absolute value of the rate of change of x greatest? When is it least?

(d) Find dx/dt when $\theta = 30°$ and $\theta = 60°$.

47. ***Flight Control*** An airplane is flying in still air with an airspeed of 275 miles per hour. If it is climbing at an angle of 18°, find the rate at which it is gaining altitude.

48. ***Security Camera*** A security camera is centered 50 feet above a 100-foot hallway (see figure). It is easiest to design the camera with a constant angular rate of rotation, but this results in a variable rate at which the images of the surveillance area are recorded. So, it is desirable to design a system with a variable rate of rotation and a constant rate of movement of the scanning beam along the hallway. Find a model for the variable rate of rotation if $|dx/dt| = 2$ feet per second.

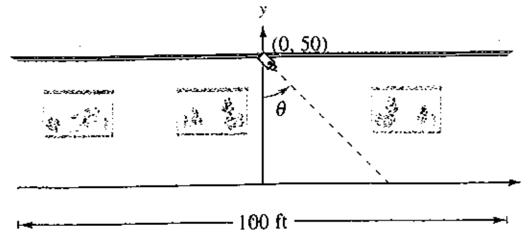


Figure for 48

49. ***Think About It*** Describe the relationship between the rate of change of y and the rate of change of x in each expression. Assume all variables and derivatives are positive.

(a) $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$ (b) $\dfrac{dy}{dt} = x(L - x)\dfrac{dx}{dt}, \quad 0 \le x \le L$

Acceleration **In Exercises 50 and 51, find the acceleration of the specified object. (*Hint:* Recall that if a variable is changing at a constant rate, its acceleration is zero.)**

50. Find the acceleration of the top of the ladder described in Exercise 25 when the base of the ladder is 7 feet from the wall.

51. Find the acceleration of the boat in Exercise 28(a) when there is a total of 13 feet of rope out.

52. ***Modeling Data*** The table shows the numbers (in millions) of single women (never married) s and married women m in the civilian work force in the United States for the years 1997 through 2005. *(Source: U.S. Bureau of Labor Statistics)*

Year	1997	1998	1999	2000	2001	2002	2003	2004	2005
s	16.5	17.1	17.6	17.8	18.0	18.2	18.4	18.6	19.2
m	33.8	33.9	34.4	35.1	35.2	35.5	36.0	35.8	35.9

(a) Use the regression capabilities of a graphing utility to find a model of the form $m(s) = as^3 + bs^2 + cs + d$ for the data, where t is the time in years, with $t = 7$ corresponding to 1997.

(b) Find dm/dt. Then use the model to estimate dm/dt for $t = 10$ if it is predicted that the number of single women in the work force will increase at the rate of 0.75 million per year.

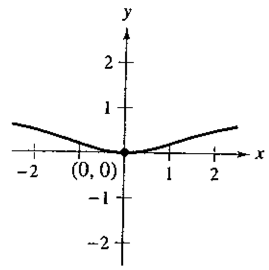
53. ***Moving Shadow*** A ball is dropped from a height of 20 meters, 12 meters away from the top of a 20-meter lamppost (see figure). The ball's shadow, caused by the light at the top of the lamppost, is moving along the level ground. How fast is the shadow moving 1 second after the ball is released? *(Submitted by Dennis Gittinger, St. Philips College, San Antonio, TX)*

3.1 Exercises

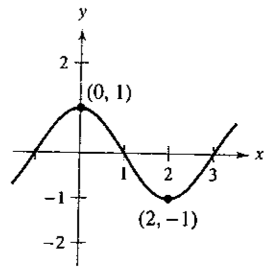
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find the value of the derivative (if it exists) at each indicated extremum.

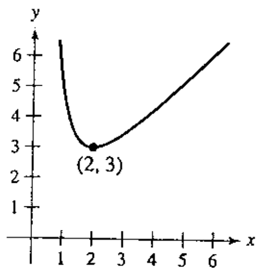
1. $f(x) = \dfrac{x^2}{x^2 + 4}$



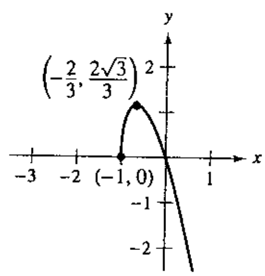
2. $f(x) = \cos \dfrac{\pi x}{2}$



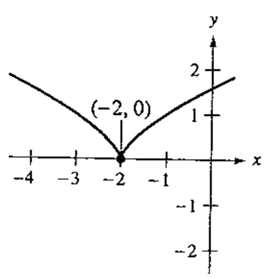
3. $g(x) = x + \dfrac{4}{x^2}$



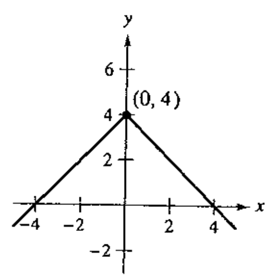
4. $f(x) = -3x\sqrt{x + 1}$



5. $f(x) = (x + 2)^{2/3}$

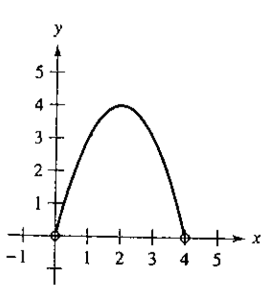


6. $f(x) = 4 - |x|$



In Exercises 7–10, approximate the critical numbers of the function shown in the graph. Determine whether the function has a relative maximum, a relative minimum, an absolute maximum, an absolute minimum, or none of these at each critical number on the interval shown.

7.



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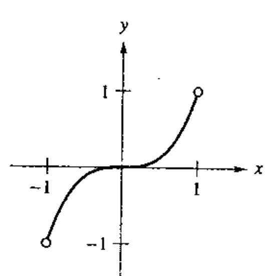

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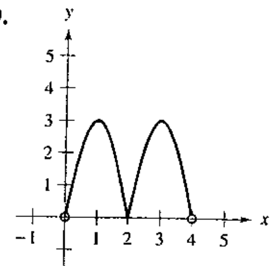

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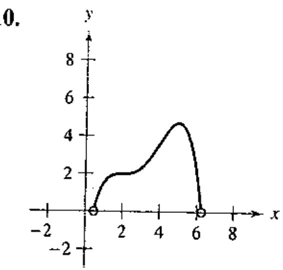


In Exercises 11–16, find any critical numbers of the function.

11. $f(x) = x^3 - 3x^2$

12. $g(x) = x^4 - 4x^2$

13. $g(t) = t\sqrt{4 - t},\ t < 3$

14. $f(x) = \dfrac{4x}{x^2 + 1}$

15. $h(x) = \sin^2 x + \cos x$, $0 < x < 2\pi$

16. $f(\theta) = 2 \sec \theta + \tan \theta$, $0 < \theta < 2\pi$

In Exercises 17–36, locate the absolute extrema of the function on the closed interval.

17. $f(x) = 3 - x,\ [-1, 2]$

18. $f(x) = \dfrac{2x + 5}{3},\ [0, 5]$

19. $g(x) = x^2 - 2x,\ [0, 4]$

20. $h(x) = -x^2 + 3x - 5,\ [-2, 1]$

21. $f(x) = x^3 - \dfrac{3}{2}x^2,\ [-1, 2]$

22. $f(x) = x^3 - 12x,\ [0, 4]$

23. $y = 3x^{2/3} - 2x,\ [-1, 1]$

24. $g(x) = \sqrt[3]{x},\ [-1, 1]$

25. $g(t) = \dfrac{t^2}{t^2 + 3},\ [-1, 1]$

26. $f(x) = \dfrac{2x}{x^2 + 1},\ [-2, 2]$

27. $h(s) = \dfrac{1}{s - 2},\ [0, 1]$

28. $h(t) = \dfrac{t}{t - 2},\ [3, 5]$

29. $y = 3 - |t - 3|,\ [-1, 5]$

30. $g(x) = \dfrac{1}{1 + |x + 1|},\ [-3, 3]$

31. $f(x) = [\![x]\!],\ [-2, 2]$

32. $h(x) = [\![2 - x]\!],\ [-2, 2]$

33. $f(x) = \cos \pi x,\ \left[0, \dfrac{1}{6}\right]$

34. $g(x) = \sec x,\ \left[-\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$

35. $y = 3 \cos x,\ [0, 2\pi]$

36. $y = \tan\left(\dfrac{\pi x}{8}\right),\ [0, 2]$

In Exercises 37–40, locate the absolute extrema of the function (if any exist) over each interval.

37. $f(x) = 2x - 3$
(a) $[0, 2]$ (b) $[0, 2)$
(c) $(0, 2]$ (d) $(0, 2)$

38. $f(x) = 5 - x$
(a) $[1, 4]$ (b) $[1, 4)$
(c) $(1, 4]$ (d) $(1, 4)$

39. $f(x) = x^2 - 2x$
(a) $[-1, 2]$ (b) $(1, 3]$
(c) $(0, 2)$ (d) $[1, 4)$

40. $f(x) = \sqrt{4 - x^2}$
(a) $[-2, 2]$ (b) $[-2, 0)$
(c) $(-2, 2)$ (d) $[1, 2)$

In Exercises 41–46, use a graphing utility to graph the function. Locate the absolute extrema of the function on the given interval.

41. $f(x) = \begin{cases} 2x + 2, & 0 \le x \le 1 \\ 4x^2, & 1 < x \le 3 \end{cases}$, $[0, 3]$

42. $f(x) = \begin{cases} 2 - x^2, & 1 \le x < 3 \\ 2 - 3x, & 3 \le x \le 5 \end{cases}$, $[1, 5]$

43. $f(x) = \dfrac{3}{x - 1}$, $(1, 4]$ **44.** $f(x) = \dfrac{2}{2 - x}$, $[0, 2)$

45. $f(x) = x^4 - 2x^3 + x + 1$, $[-1, 3]$

46. $f(x) = \sqrt{x} + \cos\dfrac{x}{2}$, $[0, 2\pi]$

CAS In Exercises 47 and 48, (a) use a computer algebra system to graph the function and approximate any absolute extrema on the given interval. (b) Use the utility to find any critical numbers, and use them to find any absolute extrema not located at the endpoints. Compare the results with those in part (a).

47. $f(x) = 3.2x^5 + 5x^3 - 3.5x$, $[0, 1]$

48. $f(x) = \dfrac{4}{3}x\sqrt{3 - x}$, $[0, 3]$

CAS In Exercises 49 and 50, use a computer algebra system to find the maximum value of $|f''(x)|$ on the closed interval. (This value is used in the error estimate for the Trapezoidal Rule, as discussed in Section 4.6.)

49. $f(x) = \sqrt{1 + x^3}$, $[0, 2]$

50. $f(x) = \dfrac{1}{x^2 + 1}$, $\left[\dfrac{1}{2}, 3\right]$

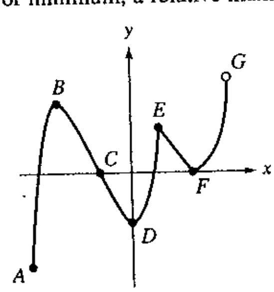
CAS In Exercises 51 and 52, use a computer algebra system to find the maximum value of $|f^{(4)}(x)|$ on the closed interval. (This value is used in the error estimate for Simpson's Rule, as discussed in Section 4.6.)

51. $f(x) = (x + 1)^{2/3}$, $[0, 2]$ **52.** $f(x) = \dfrac{1}{x^2 + 1}$, $[-1, 1]$

53. ***Writing*** Write a short paragraph explaining why a continuous function on an open interval may not have a maximum or minimum. Illustrate your explanation with a sketch of the graph of such a function.

CAPSTONE

54. Decide whether each labeled point is an absolute maximum or minimum, a relative maximum or minimum, or neither.



WRITING ABOUT CONCEPTS

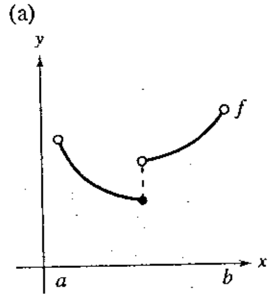
In Exercises 55 and 56, graph a function on the interval $[-2, 5]$ having the given characteristics.

55. Absolute maximum at $x = -2$, absolute minimum at $x = 1$, relative maximum at $x = 3$

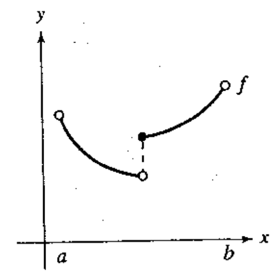
56. Relative minimum at $x = -1$, critical number (but no extremum) at $x = 0$, absolute maximum at $x = 2$, absolute minimum at $x = 5$

In Exercises 57–60, determine from the graph whether f has a minimum in the open interval (a, b).

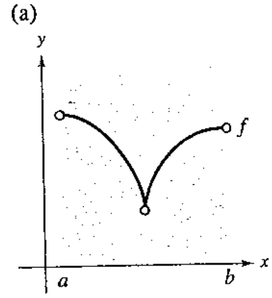
57. (a)



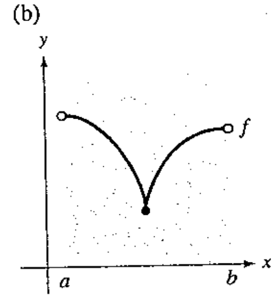
(b)



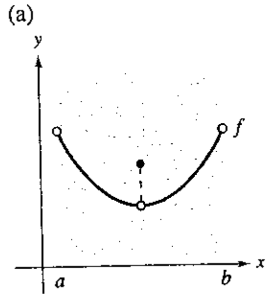
58. (a)



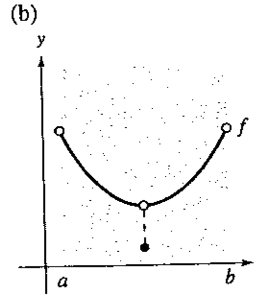
(b)



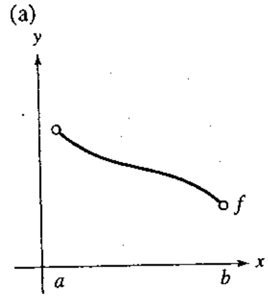
59. (a)



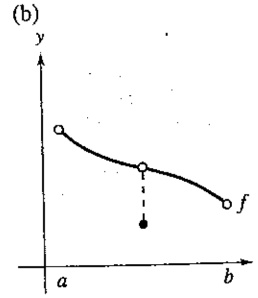
(b)



60. (a)



(b)

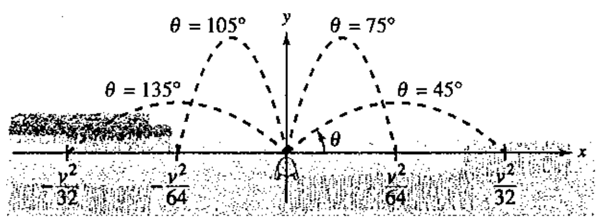


61. ***Power*** The formula for the power output P of a battery is $P = VI - RI^2$, where V is the electromotive force in volts, R is the resistance in ohms, and I is the current in amperes. Find the current that corresponds to a maximum value of P in a battery for which $V = 12$ volts and $R = 0.5$ ohm. Assume that a 15-ampere fuse bounds the output in the interval $0 \le I \le 15$. Could the power output be increased by replacing the 15-ampere fuse with a 20-ampere fuse? Explain.

62. ***Lawn Sprinkler*** A lawn sprinkler is constructed in such a way that $d\theta/dt$ is constant, where θ ranges between 45° and 135° (see figure). The distance the water travels horizontally is

$$x = \frac{v^2 \sin 2\theta}{32}, \quad 45° \le \theta \le 135°$$

where v is the speed of the water. Find dx/dt and explain why this lawn sprinkler does not water evenly. What part of the lawn receives the most water?



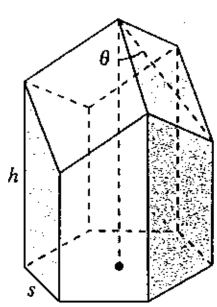
Water sprinkler: $45° \le \theta \le 135°$

FOR FURTHER INFORMATION For more information on the "calculus of lawn sprinklers," see the article "Design of an Oscillating Sprinkler" by Bart Braden in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com*.

63. ***Honeycomb*** The surface area of a cell in a honeycomb is

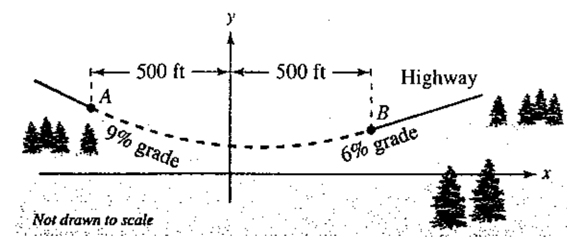
$$S = 6hs + \frac{3s^2}{2}\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right)$$

where h and s are positive constants and θ is the angle at which the upper faces meet the altitude of the cell (see figure). Find the angle θ $(\pi/6 \le \theta \le \pi/2)$ that minimizes the surface area S.



FOR FURTHER INFORMATION For more information on the geometric structure of a honeycomb cell, see the article "The Design of Honeycombs" by Anthony L. Peressini in UMAP Module 502, published by COMAP, Inc., Suite 210, 57 Bedford Street, Lexington, MA.

64. ***Highway Design*** In order to build a highway, it is necessary to fill a section of a valley where the grades (slopes) of the sides are 9% and 6% (see figure). The top of the filled region will have the shape of a parabolic arc that is tangent to the two slopes at the points A and B. The horizontal distances from A to the y-axis and from B to the y-axis are both 500 feet.



(a) Find the coordinates of A and B.

(b) Find a quadratic function $y = ax^2 + bx + c$, $-500 \le x \le 500$, that describes the top of the filled region.

(c) Construct a table giving the depths d of the fill for $x = -500, -400, -300, -200, -100, 0, 100, 200, 300, 400$, and 500.

(d) What will be the lowest point on the completed highway? Will it be directly over the point where the two hillsides come together?

True or False? **In Exercises 65–68, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

65. The maximum of a function that is continuous on a closed interval can occur at two different values in the interval.

66. If a function is continuous on a closed interval, then it must have a minimum on the interval.

67. If $x = c$ is a critical number of the function f, then it is also a critical number of the function $g(x) = f(x) + k$, where k is a constant.

68. If $x = c$ is a critical number of the function f, then it is also a critical number of the function $g(x) = f(x - k)$, where k is a constant.

69. Let the function f be differentiable on an interval I containing c. If f has a maximum value at $x = c$, show that $-f$ has a minimum value at $x = c$.

70. Consider the cubic function $f(x) = ax^3 + bx^2 + cx + d$, where $a \ne 0$. Show that f can have zero, one, or two critical numbers and give an example of each case.

PUTNAM EXAM CHALLENGE

71. Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region $R = \{(x, y);\ 0 \le x \le a,\ 0 \le y \le f(x)\}$ has perimeter k units and area k square units for some real number k.

3.2 Exercises

In Exercises 1–4, explain why Rolle's Theorem does not apply to the function even though there exist a and b such that $f(a) = f(b)$.

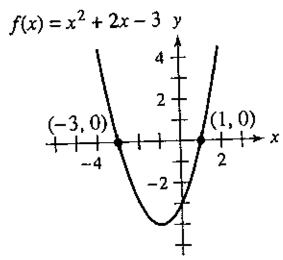
1. $f(x) = \left|\frac{1}{x}\right|$, $[-1, 1]$
2. $f(x) = \cot \frac{x}{2}$, $[\pi, 3\pi]$
3. $f(x) = 1 - |x - 1|$, $[0, 2]$
4. $f(x) = \sqrt{(2 - x^{2/3})^3}$, $[-1, 1]$

In Exercises 5–8, find the two x-intercepts of the function f and show that $f'(x) = 0$ at some point between the two x-intercepts.

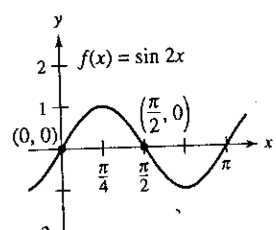
5. $f(x) = x^2 - x - 2$
6. $f(x) = x(x - 3)$
7. $f(x) = x\sqrt{x + 4}$
8. $f(x) = -3x\sqrt{x + 1}$

Rolle's Theorem **In Exercises 9 and 10, the graph of f is shown. Apply Rolle's Theorem and find all values of c such that $f'(c) = 0$ at some point between the labeled intercepts.**

9.



10.



In Exercises 11–24, determine whether Rolle's Theorem can be applied to f on the closed interval $[a, b]$. If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$. If Rolle's Theorem cannot be applied, explain why not.

11. $f(x) = -x^2 + 3x$, $[0, 3]$
12. $f(x) = x^2 - 5x + 4$, $[1, 4]$
13. $f(x) = (x - 1)(x - 2)(x - 3)$, $[1, 3]$
14. $f(x) = (x - 3)(x + 1)^2$, $[-1, 3]$
15. $f(x) = x^{2/3} - 1$, $[-8, 8]$
16. $f(x) = 3 - |x - 3|$, $[0, 6]$
17. $f(x) = \dfrac{x^2 - 2x - 3}{x + 2}$, $[-1, 3]$
18. $f(x) = \dfrac{x^2 - 1}{x}$, $[-1, 1]$
19. $f(x) = \sin x$, $[0, 2\pi]$
20. $f(x) = \cos x$, $[0, 2\pi]$
21. $f(x) = \dfrac{6x}{\pi} - 4\sin^2 x$, $\left[0, \dfrac{\pi}{6}\right]$
22. $f(x) = \cos 2x$, $[-\pi, \pi]$
23. $f(x) = \tan x$, $[0, \pi]$
24. $f(x) = \sec x$, $[\pi, 2\pi]$

In Exercises 25–28, use a graphing utility to graph the func on the closed interval $[a, b]$. Determine whether Rol Theorem can be applied to f on the interval and, if so, finc values of c in the open interval (a, b) such that $f'(c) = 0$.

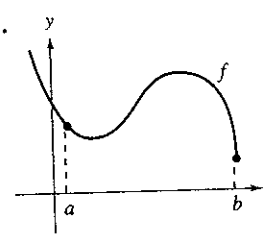
25. $f(x) = |x| - 1$, $[-1, 1]$
26. $f(x) = x - x^{1/3}$, $[0, 1$
27. $f(x) = x - \tan \pi x$, $\left[-\frac{1}{4}, \frac{1}{4}\right]$
28. $f(x) = \dfrac{x}{2} - \sin \dfrac{\pi x}{6}$, $[-1, 0]$

29. *Vertical Motion* The height of a ball t seconds after thrown upward from a height of 6 feet and with an i velocity of 48 feet per second is $f(t) = -16t^2 + 48t + 6$.
    (a) Verify that $f(1) = f(2)$.
    (b) According to Rolle's Theorem, what must the velocity some time in the interval $(1, 2)$? Find that time.

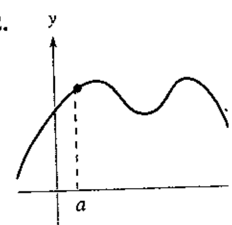
30. *Reorder Costs* The ordering and transportation cost (components used in a manufacturing process is approxir by $C(x) = 10\left(\dfrac{1}{x} + \dfrac{x}{x + 3}\right)$, where C is measured in thou of dollars and x is the order size in hundreds.
    (a) Verify that $C(3) = C(6)$.
    (b) According to Rolle's Theorem, the rate of change (cost must be 0 for some order size in the interval (Find that order size.

In Exercises 31 and 32, copy the graph and sketch the s line to the graph through the points $(a, f(a))$ and $(b, f(b))$. sketch any tangent lines to the graph for each value guaranteed by the Mean Value Theorem. To print an enl copy of the graph, go to the website *www.mathgraphs.com*

31.

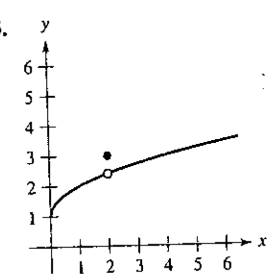


32.

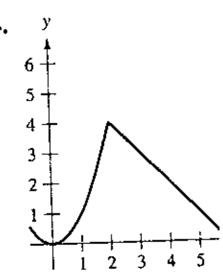


Writing **In Exercises 33–36, explain why the Mean Theorem does not apply to the function f on the interval**

33.



34.



35. $f(x) = \dfrac{1}{x - 3}$
36. $f(x) = |x - 3|$

37. ***Mean Value Theorem*** Consider the graph of the function $f(x) = -x^2 + 5$. (a) Find the equation of the secant line joining the points $(-1, 4)$ and $(2, 1)$. (b) Use the Mean Value Theorem to determine a point c in the interval $(-1, 2)$ such that the tangent line at c is parallel to the secant line. (c) Find the equation of the tangent line through c. (d) Then use a graphing utility to graph f, the secant line, and the tangent line.

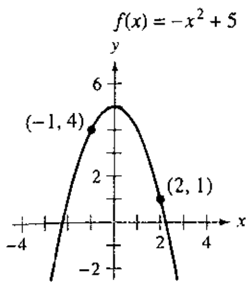


Figure for 37

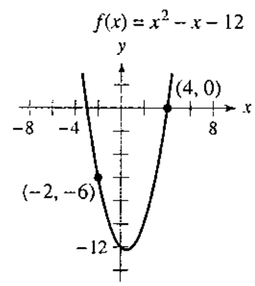


Figure for 38

38. ***Mean Value Theorem*** Consider the graph of the function $f(x) = x^2 - x - 12$. (a) Find the equation of the secant line joining the points $(-2, -6)$ and $(4, 0)$. (b) Use the Mean Value Theorem to determine a point c in the interval $(-2, 4)$ such that the tangent line at c is parallel to the secant line. (c) Find the equation of the tangent line through c. (d) Then use a graphing utility to graph f, the secant line, and the tangent line.

In Exercises 39–48, determine whether the Mean Value Theorem can be applied to f on the closed interval $[a, b]$. If the Mean Value Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. If the Mean Value Theorem cannot be applied, explain why not.

39. $f(x) = x^2, \quad [-2, 1]$

40. $f(x) = x^3, \quad [0, 1]$

41. $f(x) = x^3 + 2x, \quad [-1, 1]$

42. $f(x) = x^4 - 8x, \quad [0, 2]$

43. $f(x) = x^{2/3}, \quad [0, 1]$

44. $f(x) = \dfrac{x + 1}{x}, \quad [-1, 2]$

45. $f(x) = |2x + 1|, \quad [-1, 3]$

46. $f(x) = \sqrt{2 - x}, \quad [-7, 2]$

47. $f(x) = \sin x, \quad [0, \pi]$

48. $f(x) = \cos x + \tan x, \quad [0, \pi]$

In Exercises 49–52, use a graphing utility to (a) graph the function f on the given interval, (b) find and graph the secant line through points on the graph of f at the endpoints of the given interval, and (c) find and graph any tangent lines to the graph of f that are parallel to the secant line.

49. $f(x) = \dfrac{x}{x + 1}, \quad \left[-\frac{1}{2}, 2\right]$

50. $f(x) = x - 2\sin x, \quad [-\pi, \pi]$

51. $f(x) = \sqrt{x}, \quad [1, 9]$

52. $f(x) = x^4 - 2x^3 + x^2, \quad [0, 6]$

53. ***Vertical Motion*** The height of an object t seconds after it is dropped from a height of 300 meters is $s(t) = -4.9t^2 + 300$.

(a) Find the average velocity of the object during the first 3 seconds.

(b) Use the Mean Value Theorem to verify that at some time during the first 3 seconds of fall the instantaneous velocity equals the average velocity. Find that time.

54. ***Sales*** A company introduces a new product for which the number of units sold S is

$$S(t) = 200\left(5 - \frac{9}{2 + t}\right)$$

where t is the time in months.

(a) Find the average rate of change of $S(t)$ during the first year.

(b) During what month of the first year does $S'(t)$ equal the average rate of change?

WRITING ABOUT CONCEPTS

55. Let f be continuous on $[a, b]$ and differentiable on (a, b). If there exists c in (a, b) such that $f'(c) = 0$, does it follow that $f(a) = f(b)$? Explain.

56. Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Also, suppose that $f(a) = f(b)$ and that c is a real number in the interval such that $f'(c) = 0$. Find an interval for the function g over which Rolle's Theorem can be applied, and find the corresponding critical number of g (k is a constant).

(a) $g(x) = f(x) + k$

(b) $g(x) = f(x - k)$

(c) $g(x) = f(kx)$

57. The function

$$f(x) = \begin{cases} 0, & x = 0 \\ 1 - x, & 0 < x \le 1 \end{cases}$$

is differentiable on $(0, 1)$ and satisfies $f(0) = f(1)$. However, its derivative is never zero on $(0, 1)$. Does this contradict Rolle's Theorem? Explain.

58. Can you find a function f such that $f(-2) = -2$, $f(2) = 6$, and $f'(x) < 1$ for all x? Why or why not?

59. ***Speed*** A plane begins its takeoff at 2:00 P.M. on a 2500-mile flight. After 5.5 hours, the plane arrives at its destination. Explain why there are at least two times during the flight when the speed of the plane is 400 miles per hour.

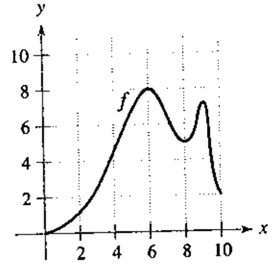
60. ***Temperature*** When an object is removed from a furnace and placed in an environment with a constant temperature of 90°F, its core temperature is 1500°F. Five hours later the core temperature is 390°F. Explain why there must exist a time in the interval when the temperature is decreasing at a rate of 222°F per hour.

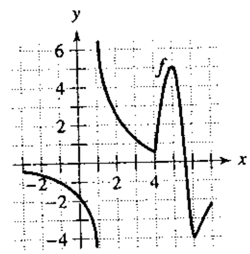
61. ***Velocity*** Two bicyclists begin a race at 8:00 A.M. They both finish the race 2 hours and 15 minutes later. Prove that at some time during the race, the bicyclists are traveling at the same velocity.

62. ***Acceleration*** At 9:13 A.M., a sports car is traveling 35 miles per hour. Two minutes later, the car is traveling 85 miles per hour. Prove that at some time during this two-minute interval, the car's acceleration is exactly 1500 miles per hour squared.

3.3 Exercises

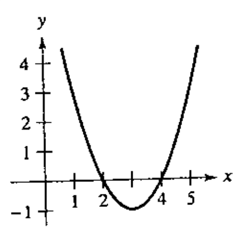
In Exercises 1 and 2, use the graph of f to find (a) the largest open interval on which f is increasing, and (b) the largest open interval on which f is decreasing.

1. 

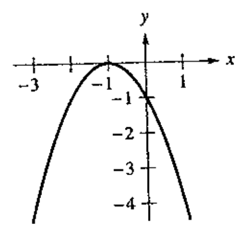
2. 

In Exercises 3–8, use the graph to estimate the open intervals on which the function is increasing or decreasing. Then find the open intervals analytically.

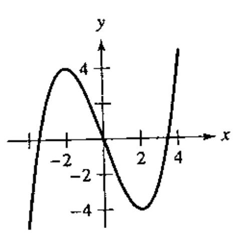
3. $f(x) = x^2 - 6x + 8$



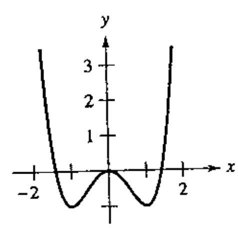
4. $y = -(x + 1)^2$



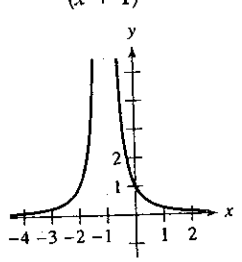
5. $y = \dfrac{x^3}{4} - 3x$



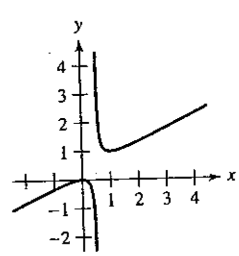
6. $f(x) = x^4 - 2x^2$



7. $f(x) = \dfrac{1}{(x + 1)^2}$



8. $y = \dfrac{x^2}{2x - 1}$



In Exercises 9–16, identify the open intervals on which the function is increasing or decreasing.

9. $g(x) = x^2 - 2x - 8$
10. $h(x) = 27x - x^3$
11. $y = x\sqrt{16 - x^2}$
12. $y = x + \dfrac{4}{x}$
13. $f(x) = \sin x - 1, \quad 0 < x < 2\pi$
14. $h(x) = \cos \dfrac{x}{2}, \quad 0 < x < 2\pi$
15. $y = x - 2 \cos x, \quad 0 < x < 2\pi$
16. $f(x) = \cos^2 x - \cos x, \quad 0 < x < 2\pi$

In Exercises 17–42, (a) find the critical numbers of f (if any), (find the open interval(s) on which the function is increasing decreasing, (c) apply the First Derivative Test to identify relative extrema, and (d) use a graphing utility to confirm yo results.

17. $f(x) = x^2 - 4x$
18. $f(x) = x^2 + 6x + 10$
19. $f(x) = -2x^2 + 4x + 3$
20. $f(x) = -(x^2 + 8x + 12)$
21. $f(x) = 2x^3 + 3x^2 - 12x$
22. $f(x) = x^3 - 6x^2 + 15$
23. $f(x) = (x - 1)^2(x + 3)$
24. $f(x) = (x + 2)^2(x - 1)$
25. $f(x) = \dfrac{x^5 - 5x}{5}$
26. $f(x) = x^4 - 32x + 4$
27. $f(x) = x^{1/3} + 1$
28. $f(x) = x^{2/3} - 4$
29. $f(x) = (x + 2)^{2/3}$
30. $f(x) = (x - 3)^{1/3}$
31. $f(x) = 5 - |x - 5|$
32. $f(x) = |x + 3| - 1$
33. $f(x) = 2x + \dfrac{1}{x}$
34. $f(x) = \dfrac{x}{x + 3}$
35. $f(x) = \dfrac{x^2}{x^2 - 9}$
36. $f(x) = \dfrac{x + 4}{x^2}$
37. $f(x) = \dfrac{x^2 - 2x + 1}{x + 1}$
38. $f(x) = \dfrac{x^2 - 3x - 4}{x - 2}$
39. $f(x) = \begin{cases} 4 - x^2, & x \le 0 \\ -2x, & x > 0 \end{cases}$
40. $f(x) = \begin{cases} 2x + 1, & x \le - \\ x^2 - 2, & x > - \end{cases}$
41. $f(x) = \begin{cases} 3x + 1, & x \le 1 \\ 5 - x^2, & x > 1 \end{cases}$
42. $f(x) = \begin{cases} -x^3 + 1, & x \\ -x^2 + 2x, & x \end{cases}$

In Exercises 43–50, consider the function on the inte $(0, 2\pi)$. For each function, (a) find the open interval(s) on wl the function is increasing or decreasing, (b) apply the F Derivative Test to identify all relative extrema, and (c) u graphing utility to confirm your results.

43. $f(x) = \dfrac{x}{2} + \cos x$
44. $f(x) = \sin x \cos x + 5$
45. $f(x) = \sin x + \cos x$
46. $f(x) = x + 2 \sin x$
47. $f(x) = \cos^2(2x)$
48. $f(x) = \sqrt{3} \sin x + \cos$
49. $f(x) = \sin^2 x + \sin x$
50. $f(x) = \dfrac{\sin x}{1 + \cos^2 x}$

CAS In Exercises 51–56, (a) use a computer algebra syste differentiate the function, (b) sketch the graphs of f and the same set of coordinate axes over the given interval, (c) the critical numbers of f in the open interval, and (d) fin interval(s) on which f' is positive and the interval(s) on wh is negative. Compare the behavior of f and the sign of f'.

51. $f(x) = 2x\sqrt{9 - x^2}, \quad [-3, 3]$

52. $f(x) = 10\left(5 - \sqrt{x^2 - 3x + 16}\right), \quad [0, 5]$

53. $f(t) = t^2 \sin t, \quad [0, 2\pi]$ 54. $f(x) = \dfrac{x}{2} + \cos\dfrac{x}{2}, \quad [0, 4\pi]$

55. $f(x) = -3 \sin\dfrac{x}{3}, \quad [0, 6\pi]$

56. $f(x) = 2 \sin 3x + 4 \cos 3x, \quad [0, \pi]$

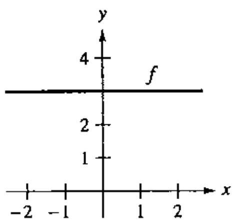
In Exercises 57 and 58, use symmetry, extrema, and zeros to sketch the graph of f. How do the functions f and g differ?

57. $f(x) = \dfrac{x^5 - 4x^3 + 3x}{x^2 - 1}, \quad g(x) = x(x^2 - 3)$

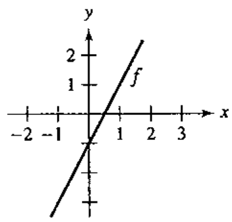
58. $f(t) = \cos^2 t - \sin^2 t, \quad g(t) = 1 - 2\sin^2 t$

Think About It **In Exercises 59–64, the graph of f is shown in the figure. Sketch a graph of the derivative of f. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

59.



60.



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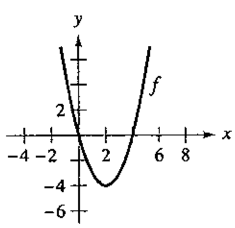

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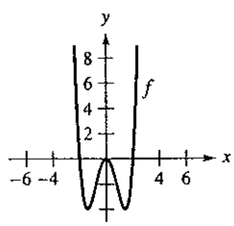

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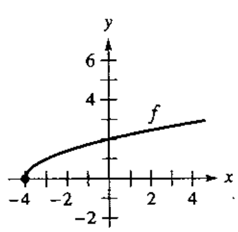
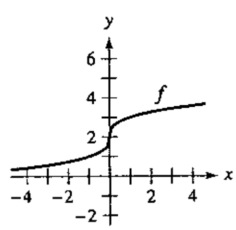
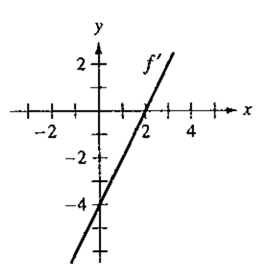


64.

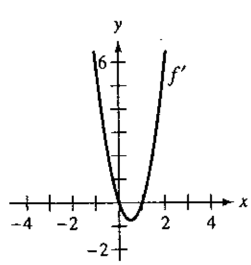


In Exercises 65–68, use the graph of f' to (a) identify the interval(s) on which f is increasing or decreasing, and (b) estimate the value(s) of x at which f has a relative maximum or minimum.

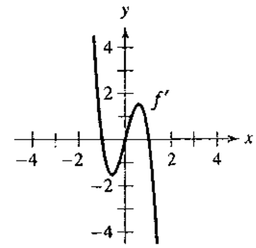
65.



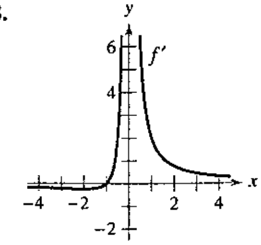
66.



67.

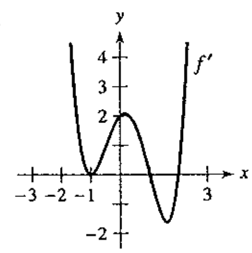


68.

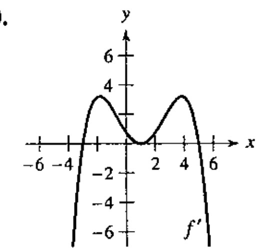


In Exercises 69 and 70, use the graph of f' to (a) identify the critical numbers of f, and (b) determine whether f has a relative maximum, a relative minimum, or neither at each critical number.

69.



70.



WRITING ABOUT CONCEPTS

In Exercises 71–76, assume that f is differentiable for all x. The signs of f' are as follows.

$f'(x) > 0$ on $(-\infty, -4)$

$f'(x) < 0$ on $(-4, 6)$

$f'(x) > 0$ on $(6, \infty)$

Supply the appropriate inequality sign for the indicated value of c.

	Function	*Sign of $g'(c)$*
71.	$g(x) = f(x) + 5$	$g'(0) \ \square\ 0$
72.	$g(x) = 3f(x) - 3$	$g'(-5) \ \square\ 0$
73.	$g(x) = -f(x)$	$g'(-6) \ \square\ 0$
74.	$g(x) = -f(x)$	$g'(0) \ \square\ 0$
75.	$g(x) = f(x - 10)$	$g'(0) \ \square\ 0$
76.	$g(x) = f(x - 10)$	$g'(8) \ \square\ 0$

77. Sketch the graph of the arbitrary function f such that

$$f'(x)\begin{cases} > 0, & x < 4 \\ \text{undefined}, & x = 4. \\ < 0, & x > 4 \end{cases}$$

CAPSTONE

78. A differentiable function f has one critical number at $x = 5$. Identify the relative extrema of f at the critical number if $f'(4) = -2.5$ and $f'(6) = 3$.

Think About It **In Exercises 79 and 80, the function f is differentiable on the indicated interval. The table shows $f'(x)$ for selected values of x. (a) Sketch the graph of f, (b) approximate the critical numbers, and (c) identify the relative extrema.**

79. f is differentiable on $[-1, 1]$

x	-1	-0.75	-0.50	-0.25
$f'(x)$	-10	-3.2	-0.5	0.8

x	0	0.25	0.50	0.75	1
$f'(x)$	5.6	3.6	-0.2	-6.7	-20.1

80. f is differentiable on $[0, \pi]$

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$f'(x)$	3.14	-0.23	-2.45	-3.11	0.69

x	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$f'(x)$	3.00	1.37	-1.14	-2.84

81. ***Rolling a Ball Bearing*** A ball bearing is placed on an inclined plane and begins to roll. The angle of elevation of the plane is θ. The distance (in meters) the ball bearing rolls in t seconds is $s(t) = 4.9(\sin\theta)t^2$.

(a) Determine the speed of the ball bearing after t seconds.

(b) Complete the table and use it to determine the value of θ that produces the maximum speed at a particular time.

θ	0	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π
$s'(t)$							

82. ***Numerical, Graphical, and Analytic Analysis*** The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is

$$C(t) = \frac{3t}{27 + t^3}, \quad t \geq 0.$$

(a) Complete the table and use it to approximate the time when the concentration is greatest.

t	0	0.5	1	1.5	2	2.5	3
$C(t)$							

(b) Use a graphing utility to graph the concentration function and use the graph to approximate the time when the concentration is greatest.

(c) Use calculus to determine analytically the time when the concentration is greatest.

83. ***Numerical, Graphical, and Analytic Analysis*** Consider the functions $f(x) = x$ and $g(x) = \sin x$ on the interval $(0, \pi)$.

(a) Complete the table and make a conjecture about which is the greater function on the interval $(0, \pi)$.

x	0.5	1	1.5	2	2.5	3
$f(x)$						
$g(x)$						

(b) Use a graphing utility to graph the functions and use th graphs to make a conjecture about which is the greate function on the interval $(0, \pi)$.

(c) Prove that $f(x) > g(x)$ on the interval $(0, \pi)$. [*Hint:* Shov that $h'(x) > 0$ where $h = f - g$.]

84. ***Numerical, Graphical, and Analytic Analysis*** Consider th functions $f(x) = x$ and $g(x) = \tan x$ on the interval $(0, \pi/2)$.

(a) Complete the table and make a conjecture about which is the greater function on the interval $(0, \pi/2)$.

x	0.25	0.5	0.75	1	1.25	1.5
$f(x)$						
$g(x)$						

(b) Use a graphing utility to graph the functions and use tl graphs to make a conjecture about which is the great function on the interval $(0, \pi/2)$.

(c) Prove that $f(x) < g(x)$ on the interval $(0, \pi/2)$. [*Hir* Show that $h'(x) > 0$, where $h = g - f$.]

85. ***Trachea Contraction*** Coughing forces the trachea (win pipe) to contract, which affects the velocity v of the air passi through the trachea. The velocity of the air during coughing $v = k(R - r)r^2$, $0 \leq r < R$, where k is a constant, R is t normal radius of the trachea, and r is the radius duri coughing. What radius will produce the maximum air velocit

86. ***Power*** The electric power P in watts in a direct-current circ with two resistors R_1 and R_2 connected in parallel is

$$P = \frac{vR_1R_2}{(R_1 + R_2)^2}$$

where v is the voltage. If v and R_1 are held constant, what resi ance R_2 produces maximum power?

87. ***Electrical Resistance*** The resistance R of a certain type resistor is $R = \sqrt{0.001T^4 - 4T + 100}$, where R is measu in ohms and the temperature T is measured in degrees Celsi

CAS (a) Use a computer algebra system to find dR/dT and critical number of the function. Determine the minim resistance for this type of resistor.

(b) Use a graphing utility to graph the function R and use graph to approximate the minimum resistance for this t of resistor.

Modeling Data The end-of-year assets of the Medicare Hospital Insurance Trust Fund (in billions of dollars) for the years 1995 through 2006 are shown.

1995: 130.3; 1996: 124.9; 1997: 115.6; 1998: 120.4;

1999: 141.4; 2000: 177.5; 2001: 208.7; 2002: 234.8;

2003: 256.0; 2004: 269.3; 2005: 285.8; 2006: 305.4

(Source: U.S. Centers for Medicare and Medicaid Services)

(a) Use the regression capabilities of a graphing utility to find a model of the form $M = at^4 + bt^3 + ct^2 + dt + e$ for the data. (Let $t = 5$ represent 1995.)

(b) Use a graphing utility to plot the data and graph the model.

(c) Find the minimum value of the model and compare the result with the actual data.

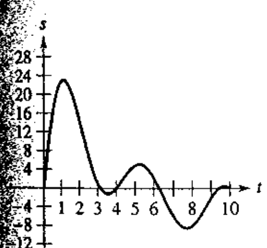
otion Along a Line **In Exercises 89–92, the function $s(t)$ scribes the motion of a particle along a line. For each function, find the velocity function of the particle at any time $t \geq 0$, identify the time interval(s) in which the particle is moving a positive direction, (c) identify the time interval(s) in which the particle is moving in a negative direction, and (d) identify the time(s) at which the particle changes direction.**

$s(t) = 6t - t^2$ **90.** $s(t) = t^2 - 7t + 10$

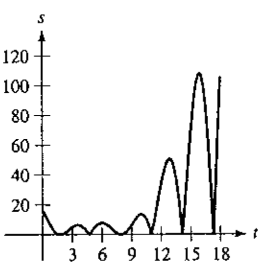
$s(t) = t^3 - 5t^2 + 4t$

$s(t) = t^3 - 20t^2 + 128t - 280$

otion Along a Line **In Exercises 93 and 94, the graph shows position of a particle moving along a line. Describe how the rticle's position changes with respect to time.**



94.



ting Polynomial Functions **In Exercises 95–98, find a nomial function**

$$= a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

has only the specified extrema. (a) Determine the minimum ree of the function and give the criteria you used in ermining the degree. (b) Using the fact that the coordinates he extrema are solution points of the function, and that the oordinates are critical numbers, determine a system of linear ations whose solution yields the coefficients of the required ction. (c) Use a graphing utility to solve the system of ations and determine the function. (d) Use a graphing ity to confirm your result graphically.

Relative minimum: (0, 0); Relative maximum: (2, 2)

Relative minimum: (0, 0); Relative maximum: (4, 1000)

Relative minima: (0, 0), (4, 0); Relative maximum: (2, 4)

Relative minimum: (1, 2); Relative maxima: (−1, 4), (3, 4)

True or False? **In Exercises 99–103, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

99. The sum of two increasing functions is increasing.

100. The product of two increasing functions is increasing.

101. Every nth-degree polynomial has $(n - 1)$ critical numbers.

102. An nth-degree polynomial has at most $(n - 1)$ critical numbers.

103. There is a relative maximum or minimum at each critical number.

104. Prove the second case of Theorem 3.5.

105. Prove the second case of Theorem 3.6.

106. Use the definitions of increasing and decreasing functions to prove that $f(x) = x^3$ is increasing on $(-\infty, \infty)$.

107. Use the definitions of increasing and decreasing functions to prove that $f(x) = 1/x$ is decreasing on $(0, \infty)$.

PUTNAM EXAM CHALLENGE

108. Find the minimum value of

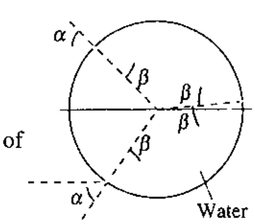
$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x.

SECTION PROJECT

Rainbows

Rainbows are formed when light strikes raindrops and is reflected and refracted, as shown in the figure. (This figure shows a cross section of a spherical raindrop.) The Law of Refraction states that $(\sin \alpha)/(\sin \beta) = k$, where $k \approx 1.33$ (for water). The angle of deflection is given by $D = \pi + 2\alpha - 4\beta$.



(a) Use a graphing utility to graph

$D = \pi + 2\alpha - 4\sin^{-1}(1/k \sin \alpha)$,

$0 \leq \alpha \leq \pi/2$.

(b) Prove that the minimum angle of deflection occurs when

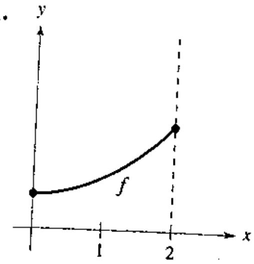
$$\cos \alpha = \sqrt{\frac{k^2 - 1}{3}}.$$

For water, what is the minimum angle of deflection, D_{min}? (The angle $\pi - D_{min}$ is called the *rainbow angle*.) What value of α produces this minimum angle? (A ray of sunlight that strikes a raindrop at this angle, α, is called a *rainbow ray*.)

FOR FURTHER INFORMATION For more information about the mathematics of rainbows, see the article "Somewhere Within the Rainbow" by Steven Janke in *The UMAP Journal*.

3.4 Exercises

In Exercises 1–4, the graph of f is shown. State the signs of f' and f'' on the interval $(0, 2)$.

1.


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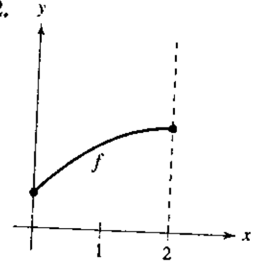
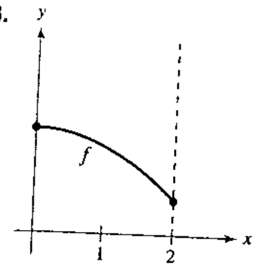


3.


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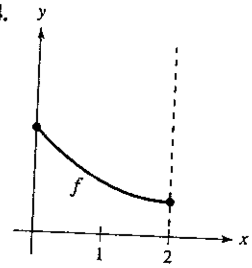


In Exercises 5–18, determine the open intervals on which the graph is concave upward or concave downward.

5. $y = x^2 - x - 2$
6. $y = -x^3 + 3x^2 - 2$
7. $g(x) = 3x^2 - x^3$
8. $h(x) = x^5 - 5x + 2$
9. $f(x) = -x^3 + 6x^2 - 9x - 1$
10. $f(x) = x^5 + 5x^4 - 40x^2$
11. $f(x) = \dfrac{24}{x^2 + 12}$
12. $f(x) = \dfrac{x^2}{x^2 + 1}$
13. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$
14. $y = \dfrac{-3x^5 + 40x^3 + 135x}{270}$
15. $g(x) = \dfrac{x^2 + 4}{4 - x^2}$
16. $h(x) = \dfrac{x^2 - 1}{2x - 1}$
17. $y = 2x - \tan x, \quad \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
18. $y = x + \dfrac{2}{\sin x}, \quad (-\pi, \pi)$

In Exercises 19–36, find the points of inflection and discuss the concavity of the graph of the function.

19. $f(x) = \frac{1}{2}x^4 + 2x^3$
20. $f(x) = -x^4 + 24x^2$
21. $f(x) = x^3 - 6x^2 + 12x$
22. $f(x) = 2x^3 - 3x^2 - 12x + 5$
23. $f(x) = \frac{1}{4}x^4 - 2x^2$
24. $f(x) = 2x^4 - 8x + 3$
25. $f(x) = x(x - 4)^3$
26. $f(x) = (x - 2)^3(x - 1)$
27. $f(x) = x\sqrt{x + 3}$
28. $f(x) = x\sqrt{9 - x}$
29. $f(x) = \dfrac{4}{x^2 + 1}$
30. $f(x) = \dfrac{x + 1}{\sqrt{x}}$
31. $f(x) = \sin \dfrac{x}{2}, \quad [0, 4\pi]$
32. $f(x) = 2 \csc \dfrac{3x}{2}, \quad (0, 2\pi)$
33. $f(x) = \sec\left(x - \dfrac{\pi}{2}\right), \quad (0, 4\pi)$
34. $f(x) = \sin x + \cos x, \quad [0, 2\pi]$
35. $f(x) = 2 \sin x + \sin 2x, \quad [0, 2\pi]$
36. $f(x) = x + 2 \cos x, \quad [0, 2\pi]$

In Exercises 37–52, find all relative extrema. Use the Second Derivative Test where applicable.

37. $f(x) = (x - 5)^2$
38. $f(x) = -(x - 5)^2$
39. $f(x) = 6x - x^2$
40. $f(x) = x^2 + 3x - 8$
41. $f(x) = x^3 - 3x^2 + 3$
42. $f(x) = x^3 - 5x^2 + 7x$
43. $f(x) = x^4 - 4x^3 + 2$
44. $f(x) = -x^4 + 4x^3 + 8x^2$
45. $g(x) = x^2(6 - x)^3$
46. $g(x) = -\frac{1}{8}(x + 2)^2(x - 4)^2$
47. $f(x) = x^{2/3} - 3$
48. $f(x) = \sqrt{x^2 + 1}$
49. $f(x) = x + \dfrac{4}{x}$
50. $f(x) = \dfrac{x}{x - 1}$
51. $f(x) = \cos x - x, \quad [0, 4\pi]$
52. $f(x) = 2 \sin x + \cos 2x, \quad [0, 2\pi]$

CAS **In Exercises 53–56, use a computer algebra system to analyze the function over the given interval. (a) Find the first and second derivatives of the function. (b) Find any relative extrema and points of inflection. (c) Graph f, f', and f'' on the same set of coordinate axes and state the relationship between the behavior of f and the signs of f' and f''.**

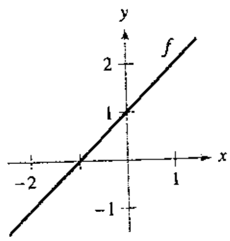
53. $f(x) = 0.2x^2(x - 3)^3, \quad [-1, 4]$
54. $f(x) = x^2\sqrt{6 - x^2}, \quad [-\sqrt{6}, \sqrt{6}]$
55. $f(x) = \sin x - \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x, \quad [0, \pi]$
56. $f(x) = \sqrt{2x}\sin x, \quad [0, 2\pi]$

WRITING ABOUT CONCEPTS

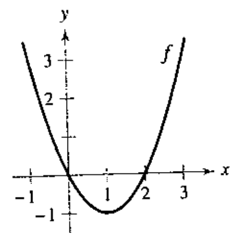
57. Consider a function f such that f' is increasing. Sketch graphs of f for (a) $f' < 0$ and (b) $f' > 0$.
58. Consider a function f such that f' is decreasing. Sketch graphs of f for (a) $f' < 0$ and (b) $f' > 0$.
59. Sketch the graph of a function f that does *not* have a point of inflection at $(c, f(c))$ even though $f''(c) = 0$.
60. S represents weekly sales of a product. What can be said of S' and S'' for each of the following statements?
    (a) The rate of change of sales is increasing.
    (b) Sales are increasing at a slower rate.
    (c) The rate of change of sales is constant.
    (d) Sales are steady.
    (e) Sales are declining, but at a slower rate.
    (f) Sales have bottomed out and have started to rise.

In Exercises 61–64, the graph of f is shown. Graph f, f', and f'' on the same set of coordinate axes. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

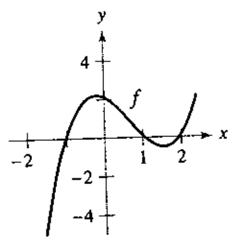
61.



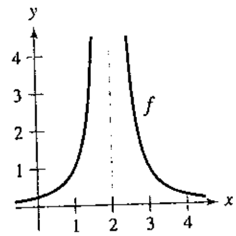
62.



63.



64.



Think About It **In Exercises 65–68, sketch the graph of a function f having the given characteristics.**

65. $f(2) = f(4) = 0$
    $f'(x) < 0$ if $x < 3$
    $f'(3)$ does not exist.
    $f'(x) > 0$ if $x > 3$
    $f''(x) < 0, x \neq 3$

66. $f(0) = f(2) = 0$
    $f'(x) > 0$ if $x < 1$
    $f'(1) = 0$
    $f'(x) < 0$ if $x > 1$
    $f''(x) < 0$

67. $f(2) = f(4) = 0$
    $f'(x) > 0$ if $x < 3$
    $f'(3)$ does not exist.
    $f'(x) < 0$ if $x > 3$
    $f''(x) > 0, x \neq 3$

68. $f(0) = f(2) = 0$
    $f'(x) < 0$ if $x < 1$
    $f'(1) = 0$
    $f'(x) > 0$ if $x > 1$
    $f''(x) > 0$

69. ***Think About It*** The figure shows the graph of f''. Sketch a graph of f. (The answer is not unique.) To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

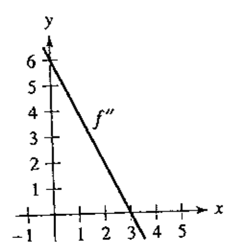


Figure for 69

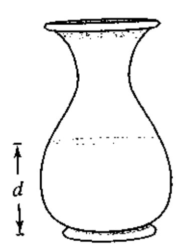


Figure for 70

CAPSTONE

70. ***Think About It*** Water is running into the vase shown in the figure at a constant rate.
    (a) Graph the depth d of water in the vase as a function of time.
    (b) Does the function have any extrema? Explain.
    (c) Interpret the inflection points of the graph of d.

71. ***Conjecture*** Consider the function $f(x) = (x - 2)^n$.
    (a) Use a graphing utility to graph f for $n = 1, 2, 3$, and 4. Use the graphs to make a conjecture about the relationship between n and any inflection points of the graph of f.
    (b) Verify your conjecture in part (a).

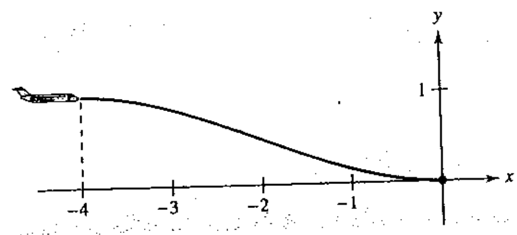
72. (a) Graph $f(x) = \sqrt[3]{x}$ and identify the inflection point.
    (b) Does $f''(x)$ exist at the inflection point? Explain.

In Exercises 73 and 74, find a, b, c, and d such that the cubic $f(x) = ax^3 + bx^2 + cx + d$ satisfies the given conditions.

73. Relative maximum: $(3, 3)$
    Relative minimum: $(5, 1)$
    Inflection point: $(4, 2)$

74. Relative maximum: $(2, 4)$
    Relative minimum: $(4, 2)$
    Inflection point: $(3, 3)$

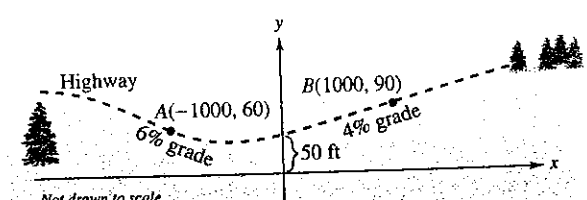
75. ***Aircraft Glide Path*** A small aircraft starts its descent from an altitude of 1 mile, 4 miles west of the runway (see figure).



    (a) Find the cubic $f(x) = ax^3 + bx^2 + cx + d$ on the interval $[-4, 0]$ that describes a smooth glide path for the landing.
    (b) The function in part (a) models the glide path of the plane. When would the plane be descending at the greatest rate?

FOR FURTHER INFORMATION For more information on this type of modeling, see the article "How Not to Land at Lake Tahoe!" by Richard Barshinger in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

76. ***Highway Design*** A section of highway connecting two hillsides with grades of 6% and 4% is to be built between two points that are separated by a horizontal distance of 2000 feet (see figure). At the point where the two hillsides come together, there is a 50-foot difference in elevation.



    (a) Design a section of highway connecting the hillsides modeled by the function $f(x) = ax^3 + bx^2 + cx + d$ $(-1000 \leq x \leq 1000)$. At the points A and B, the slope of the model must match the grade of the hillside.
    (b) Use a graphing utility to graph the model.
    (c) Use a graphing utility to graph the derivative of the model.
    (d) Determine the grade at the steepest part of the transitional section of the highway.

77. ***Beam Deflection*** The deflection D of a beam of length L is $D = 2x^4 - 5Lx^3 + 3L^2x^2$, where x is the distance from one end of the beam. Find the value of x that yields the maximum deflection.

78. ***Specific Gravity*** A model for the specific gravity of water S is

$$S = \frac{5.755}{10^8}T^3 - \frac{8.521}{10^6}T^2 + \frac{6.540}{10^5}T + 0.99987,\ 0 < T < 25$$

where T is the water temperature in degrees Celsius.

CAS (a) Use a computer algebra system to find the coordinates of the maximum value of the function.

(b) Sketch a graph of the function over the specified domain. (Use a setting in which $0.996 \le S \le 1.001$.)

(c) Estimate the specific gravity of water when $T = 20°$.

79. ***Average Cost*** A manufacturer has determined that the total cost C of operating a factory is $C = 0.5x^2 + 15x + 5000$, where x is the number of units produced. At what level of production will the average cost per unit be minimized? (The average cost per unit is C/x.)

80. ***Inventory Cost*** The total cost C of ordering and storing x units is $C = 2x + (300{,}000/x)$. What order size will produce a minimum cost?

81. ***Sales Growth*** The annual sales S of a new product are given by $S = \frac{5000t^2}{8 + t^2}$, $0 \le t \le 3$, where t is time in years.

(a) Complete the table. Then use it to estimate when the annual sales are increasing at the greatest rate.

t	0.5	1	1.5	2	2.5	3
S						

(b) Use a graphing utility to graph the function S. Then use the graph to estimate when the annual sales are increasing at the greatest rate.

(c) Find the exact time when the annual sales are increasing at the greatest rate.

82. ***Modeling Data*** The average typing speed S (in words per minute) of a typing student after t weeks of lessons is shown in the table.

t	5	10	15	20	25	30
S	38	56	79	90	93	94

A model for the data is $S = \frac{100t^2}{65 + t^2}$, $t > 0$.

(a) Use a graphing utility to plot the data and graph the model.

(b) Use the second derivative to determine the concavity of S. Compare the result with the graph in part (a).

(c) What is the sign of the first derivative for $t > 0$? By combining this information with the concavity of the model, what inferences can be made about the typing speed as t increases?

Linear and Quadratic Approximations **In Exercises 83–86, use a graphing utility to graph the function. Then graph the linear and quadratic approximations**

$$P_1(x) = f(a) + f'(a)(x - a)$$

and

$$P_2(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2$$

in the same viewing window. Compare the values of f, P_1, and P_2 and their first derivatives at $x = a$. How do the approximations change as you move farther away from $x = a$?

Function	*Value of a*
83. $f(x) = 2(\sin x + \cos x)$	$a = \frac{\pi}{4}$
84. $f(x) = 2(\sin x + \cos x)$	$a = 0$
85. $f(x) = \sqrt{1 - x}$	$a = 0$
86. $f(x) = \frac{\sqrt{x}}{x - 1}$	$a = 2$

87. Use a graphing utility to graph $y = x\sin(1/x)$. Show that the graph is concave downward to the right of $x = 1/\pi$.

88. Show that the point of inflection of $f(x) = x(x - 6)^2$ lies midway between the relative extrema of f.

89. Prove that every cubic function with three distinct real zeros has a point of inflection whose x-coordinate is the average of the three zeros.

90. Show that the cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ has exactly one point of inflection (x_0, y_0), where

$$x_0 = \frac{-b}{3a} \quad \text{and} \quad y_0 = \frac{2b^3}{27a^2} - \frac{bc}{3a} + d.$$

Use this formula to find the point of inflection of $p(x) = x^3 - 3x^2 + 2$.

True or False? **In Exercises 91–94, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

91. The graph of every cubic polynomial has precisely one point of inflection.

92. The graph of $f(x) = 1/x$ is concave downward for $x < 0$ and concave upward for $x > 0$, and thus it has a point of inflection at $x = 0$.

93. If $f'(c) > 0$, then f is concave upward at $x = c$.

94. If $f''(2) = 0$, then the graph of f must have a point of inflection at $x = 2$.

In Exercises 95 and 96, let f and g represent differentiable functions such that $f'' \neq 0$ and $g'' \neq 0$.

95. Show that if f and g are concave upward on the interval (a, b), then $f + g$ is also concave upward on (a, b).

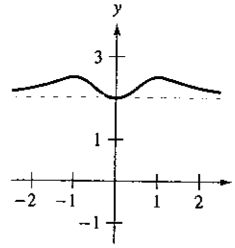
96. Prove that if f and g are positive, increasing, and concave upward on the interval (a, b), then fg is also concave upward on (a, b).

3.5 Exercises

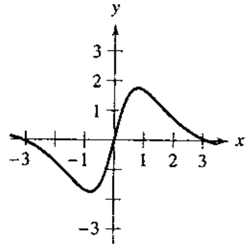
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, match the function with one of the graphs [(a), (b), (c), (d), (e), or (f)] using horizontal asymptotes as an aid.

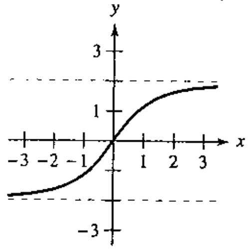
(a)



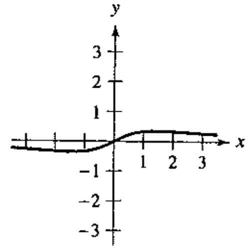
(b)



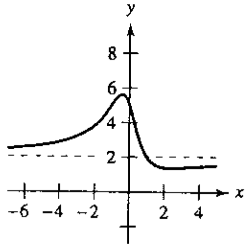
(c)



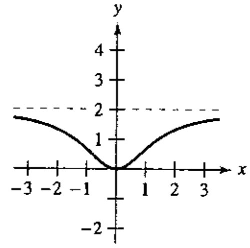
(d)



(e)



(f)



1. $f(x) = \dfrac{2x^2}{x^2 + 2}$

2. $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$

3. $f(x) = \dfrac{x}{x^2 + 2}$

4. $f(x) = 2 + \dfrac{x^2}{x^4 + 1}$

5. $f(x) = \dfrac{4 \sin x}{x^2 + 1}$

6. $f(x) = \dfrac{2x^2 - 3x + 5}{x^2 + 1}$

Numerical and Graphical Analysis **In Exercises 7–12, use a graphing utility to complete the table and estimate the limit as x approaches infinity. Then use a graphing utility to graph the function and estimate the limit graphically.**

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

7. $f(x) = \dfrac{4x + 3}{2x - 1}$

8. $f(x) = \dfrac{2x^2}{x + 1}$

9. $f(x) = \dfrac{-6x}{\sqrt{4x^2 + 5}}$

10. $f(x) = \dfrac{20x}{\sqrt{9x^2 - 1}}$

11. $f(x) = 5 - \dfrac{1}{x^2 + 1}$

12. $f(x) = 4 + \dfrac{3}{x^2 + 2}$

In Exercises 13 and 14, find $\lim_{x\to\infty} h(x)$, if possible.

13. $f(x) = 5x^3 - 3x^2 + 10x$
   (a) $h(x) = \dfrac{f(x)}{x^2}$
   (b) $h(x) = \dfrac{f(x)}{x^3}$
   (c) $h(x) = \dfrac{f(x)}{x^4}$

14. $f(x) = -4x^2 + 2x - 5$
   (a) $h(x) = \dfrac{f(x)}{x}$
   (b) $h(x) = \dfrac{f(x)}{x^2}$
   (c) $h(x) = \dfrac{f(x)}{x^3}$

In Exercises 15–18, find each limit, if possible.

15. (a) $\lim_{x\to\infty} \dfrac{x^2 + 2}{x^3 - 1}$
   (b) $\lim_{x\to\infty} \dfrac{x^2 + 2}{x^2 - 1}$
   (c) $\lim_{x\to\infty} \dfrac{x^2 + 2}{x - 1}$

16. (a) $\lim_{x\to\infty} \dfrac{3 - 2x}{3x^3 - 1}$
   (b) $\lim_{x\to\infty} \dfrac{3 - 2x}{3x - 1}$
   (c) $\lim_{x\to\infty} \dfrac{3 - 2x^2}{3x - 1}$

17. (a) $\lim_{x\to\infty} \dfrac{5 - 2x^{3/2}}{3x^2 - 4}$
   (b) $\lim_{x\to\infty} \dfrac{5 - 2x^{3/2}}{3x^{3/2} - 4}$
   (c) $\lim_{x\to\infty} \dfrac{5 - 2x^{3/2}}{3x - 4}$

18. (a) $\lim_{x\to\infty} \dfrac{5x^{3/2}}{4x^2 + 1}$
   (b) $\lim_{x\to\infty} \dfrac{5x^{3/2}}{4x^{3/2} + 1}$
   (c) $\lim_{x\to\infty} \dfrac{5x^{3/2}}{4\sqrt{x} + 1}$

In Exercises 19–38, find the limit.

19. $\lim_{x\to\infty} \left(4 + \dfrac{3}{x}\right)$

20. $\lim_{x\to-\infty} \left(\dfrac{5}{x} - \dfrac{x}{3}\right)$

21. $\lim_{x\to\infty} \dfrac{2x - 1}{3x + 2}$

22. $\lim_{x\to\infty} \dfrac{x^2 + 3}{2x^2 - 1}$

23. $\lim_{x\to\infty} \dfrac{x}{x^2 - 1}$

24. $\lim_{x\to\infty} \dfrac{5x^3 + 1}{10x^3 - 3x^2 + 7}$

25. $\lim_{x\to-\infty} \dfrac{5x^2}{x + 3}$

26. $\lim_{x\to-\infty} \left(\dfrac{1}{2}x - \dfrac{4}{x^2}\right)$

27. $\lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2 - x}}$

28. $\lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2 + 1}}$

29. $\lim_{x\to-\infty} \dfrac{2x + 1}{\sqrt{x^2 - x}}$

30. $\lim_{x\to-\infty} \dfrac{-3x + 1}{\sqrt{x^2 + x}}$

31. $\lim_{x\to\infty} \dfrac{\sqrt{x^2 - 1}}{2x - 1}$

32. $\lim_{x\to-\infty} \dfrac{\sqrt{x^4 - 1}}{x^3 - 1}$

33. $\lim_{x\to\infty} \dfrac{x + 1}{(x^2 + 1)^{1/3}}$

34. $\lim_{x\to-\infty} \dfrac{2x}{(x^6 - 1)^{1/3}}$

35. $\lim_{x\to\infty} \dfrac{1}{2x + \sin x}$

36. $\lim_{x\to\infty} \cos \dfrac{1}{x}$

37. $\lim_{x\to\infty} \dfrac{\sin 2x}{x}$

38. $\lim_{x\to\infty} \dfrac{x - \cos x}{x}$

In Exercises 39–42, use a graphing utility to graph the function and identify any horizontal asymptotes.

39. $f(x) = \dfrac{|x|}{x+1}$

40. $f(x) = \dfrac{|3x+2|}{x-2}$

41. $f(x) = \dfrac{3x}{\sqrt{x^2+2}}$

42. $f(x) = \dfrac{\sqrt{9x^2-2}}{2x+1}$

In Exercises 43 and 44, find the limit. (*Hint:* Let $x = 1/t$ and find the limit as $t \to 0^+$.)

43. $\displaystyle\lim_{x\to\infty} x \sin \frac{1}{x}$

44. $\displaystyle\lim_{x\to\infty} x \tan \frac{1}{x}$

In Exercises 45–48, find the limit. (*Hint:* Treat the expression as a fraction whose denominator is 1, and rationalize the numerator.) Use a graphing utility to verify your result.

45. $\displaystyle\lim_{x\to-\infty} \left(x + \sqrt{x^2+3}\right)$

46. $\displaystyle\lim_{x\to\infty} \left(x - \sqrt{x^2+x}\right)$

47. $\displaystyle\lim_{x\to-\infty} \left(3x + \sqrt{9x^2-x}\right)$

48. $\displaystyle\lim_{x\to\infty} \left(4x - \sqrt{16x^2-x}\right)$

Numerical, Graphical, and Analytic Analysis **In Exercises 49–52, use a graphing utility to complete the table and estimate the limit as x approaches infinity. Then use a graphing utility to graph the function and estimate the limit. Finally, find the limit analytically and compare your results with the estimates.**

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

49. $f(x) = x - \sqrt{x(x-1)}$

50. $f(x) = x^2 - x\sqrt{x(x-1)}$

51. $f(x) = x \sin \dfrac{1}{2x}$

52. $f(x) = \dfrac{x+1}{x\sqrt{x}}$

WRITING ABOUT CONCEPTS

In Exercises 53 and 54, describe in your own words what the statement means.

53. $\displaystyle\lim_{x\to\infty} f(x) = 4$

54. $\displaystyle\lim_{x\to-\infty} f(x) = 2$

55. Sketch a graph of a differentiable function f that satisfies the following conditions and has $x = 2$ as its only critical number.

$f'(x) < 0$ for $x < 2$ $\qquad f'(x) > 0$ for $x > 2$

$\displaystyle\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 6$

56. Is it possible to sketch a graph of a function that satisfies the conditions of Exercise 55 and has *no* points of inflection? Explain.

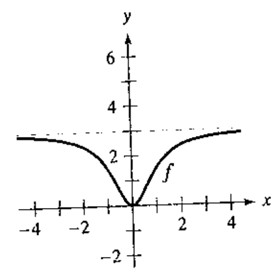
57. If f is a continuous function such that $\displaystyle\lim_{x\to\infty} f(x) = 5$, find, if possible, $\displaystyle\lim_{x\to-\infty} f(x)$ for each specified condition.

(a) The graph of f is symmetric with respect to the y-axis.

(b) The graph of f is symmetric with respect to the origin.

CAPSTONE

58. The graph of a function f is shown below. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



(a) Sketch f'.

(b) Use the graphs to estimate $\displaystyle\lim_{x\to\infty} f(x)$ and $\displaystyle\lim_{x\to\infty} f'(x)$.

(c) Explain the answers you gave in part (b).

In Exercises 59–76, sketch the graph of the equation us extrema, intercepts, symmetry, and asymptotes. Then us graphing utility to verify your result.

59. $y = \dfrac{x}{1-x}$

60. $y = \dfrac{x-4}{x-3}$

61. $y = \dfrac{x+1}{x^2-4}$

62. $y = \dfrac{2x}{9-x^2}$

63. $y = \dfrac{x^2}{x^2+16}$

64. $y = \dfrac{x^2}{x^2-16}$

65. $y = \dfrac{2x^2}{x^2-4}$

66. $y = \dfrac{2x^2}{x^2+4}$

67. $xy^2 = 9$

68. $x^2y = 9$

69. $y = \dfrac{3x}{1-x}$

70. $y = \dfrac{3x}{1-x^2}$

71. $y = 2 - \dfrac{3}{x^2}$

72. $y = 1 + \dfrac{1}{x}$

73. $y = 3 + \dfrac{2}{x}$

74. $y = 4\left(1 - \dfrac{1}{x^2}\right)$

75. $y = \dfrac{x^3}{\sqrt{x^2-4}}$

76. $y = \dfrac{x}{\sqrt{x^2-4}}$

CAS In Exercises 77–84, use a computer algebra system to ai the graph of the function. Label any extrema and/or asym that exist.

77. $f(x) = 9 - \dfrac{5}{x^2}$

78. $f(x) = \dfrac{1}{x^2 - x -}$

79. $f(x) = \dfrac{x-2}{x^2-4x+3}$

80. $f(x) = \dfrac{x+1}{x^2+x+}$

81. $f(x) = \dfrac{3x}{\sqrt{4x^2+1}}$

82. $g(x) = \dfrac{2x}{\sqrt{3x^2+}}$

83. $g(x) = \sin\left(\dfrac{x}{x-2}\right), \quad x > 3$

84. $f(x) = \dfrac{2 \sin 2x}{x}$

In Exercises 85 and 86, (a) use a graphing utility to graph f and g in the same viewing window, (b) verify algebraically that f and g represent the same function, and (c) zoom out sufficiently far so that the graph appears as a line. What equation does this line appear to have? (Note that the points at which the function is not continuous are not readily seen when you zoom out.)

85. $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x - 3)}$

$g(x) = x + \dfrac{2}{x(x - 3)}$

86. $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$

$g(x) = -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2}$

87. ***Engine Efficiency*** The efficiency of an internal combustion engine is

$$\text{Efficiency (\%)} = 100\left[1 - \frac{1}{(v_1/v_2)^c}\right]$$

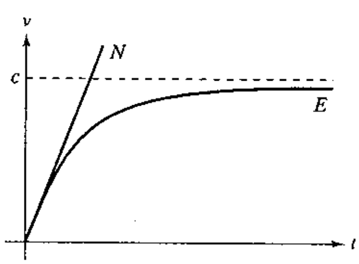
where v_1/v_2 is the ratio of the uncompressed gas to the compressed gas and c is a positive constant dependent on the engine design. Find the limit of the efficiency as the compression ratio approaches infinity.

88. ***Average Cost*** A business has a cost of $C = 0.5x + 500$ for producing x units. The average cost per unit is

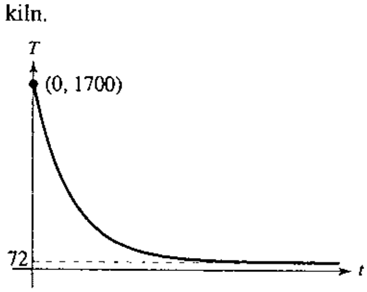
$$\overline{C} = \frac{C}{x}.$$

Find the limit of $\overline{C}$ as x approaches infinity.

89. ***Physics*** Newton's First Law of Motion and Einstein's Special Theory of Relativity differ concerning a particle's behavior as its velocity approaches the speed of light c. In the graph, functions N and E represent the velocity v, with respect to time t, of a particle accelerated by a constant force as predicted by Newton and Einstein. Write limit statements that describe these two theories.



90. ***Temperature*** The graph shows the temperature T, in degrees Fahrenheit, of molten glass t seconds after it is removed from a kiln.



(a) Find $\lim_{t \to 0^+} T$. What does this limit represent?

(b) Find $\lim_{t \to \infty} T$. What does this limit represent?

(c) Will the temperature of the glass ever actually reach room temperature? Why?

91. ***Modeling Data*** The table shows the world record times for the mile run, where t represents the year, with $t = 0$ corresponding to 1900, and y is the time in minutes and seconds.

t	23	33	45	54	58
y	4:10.4	4:07.6	4:01.3	3:59.4	3:54.5

t	66	79	85	99
y	3:51.3	3:48.9	3:46.3	3:43.1

A model for the data is

$$y = \frac{3.351t^2 + 42.461t - 543.730}{t^2}$$

where the seconds have been changed to decimal parts of a minute.

(a) Use a graphing utility to plot the data and graph the model.

(b) Does there appear to be a limiting time for running 1 mile? Explain.

92. ***Modeling Data*** The average typing speeds S (in words per minute) of a typing student after t weeks of lessons are shown in the table.

t	5	10	15	20	25	30
S	28	56	79	90	93	94

A model for the data is $S = \dfrac{100t^2}{65 + t^2}$, $t > 0$.

(a) Use a graphing utility to plot the data and graph the model.

(b) Does there appear to be a limiting typing speed? Explain.

93. ***Modeling Data*** A heat probe is attached to the heat exchanger of a heating system. The temperature T (in degrees Celsius) is recorded t seconds after the furnace is started. The results for the first 2 minutes are recorded in the table.

t	0	15	30	45	60
T	25.2°	36.9°	45.5°	51.4°	56.0°

t	75	90	105	120
T	59.6°	62.0°	64.0°	65.2°

(a) Use the regression capabilities of a graphing utility to find a model of the form $T_1 = at^2 + bt + c$ for the data.

(b) Use a graphing utility to graph T_1.

(c) A rational model for the data is $T_2 = \dfrac{1451 + 86t}{58 + t}$. Use a graphing utility to graph T_2.

(d) Find $T_1(0)$ and $T_2(0)$.

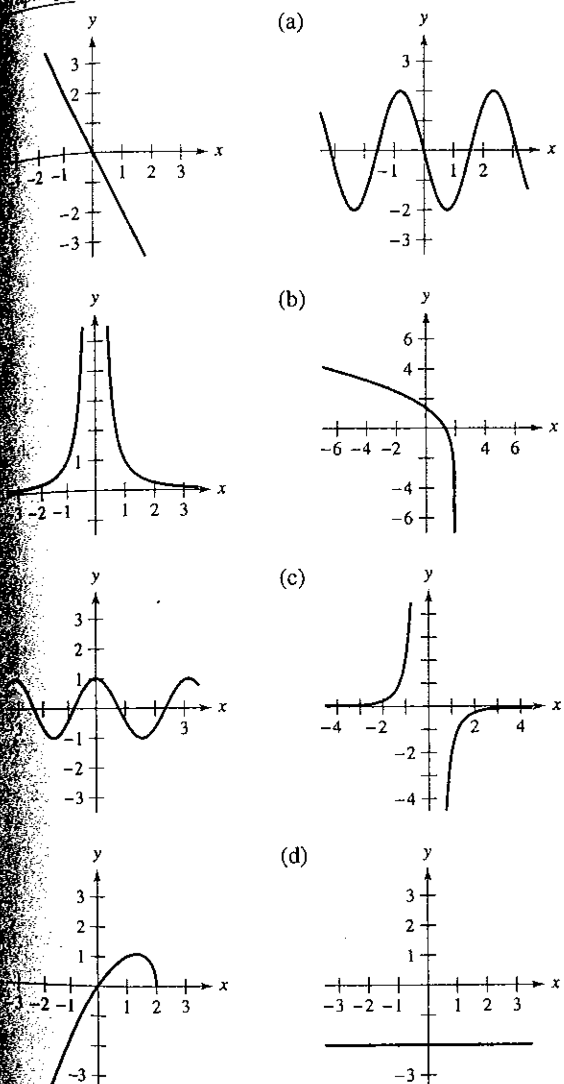
(e) Find $\lim_{t \to \infty} T_2$.

(f) Interpret the result in part (e) in the context of the problem. Is it possible to do this type of analysis using T_1? Explain.

6 Exercises

rcises 1–4, match the graph of *f* in the left column with its derivative in the right column.

ph of f *Graph of f′*



xercises 5–32, analyze and sketch a graph of the function. el any intercepts, relative extrema, points of inflection, and mptotes. Use a graphing utility to verify your results.

$y = \frac{1}{x-2} - 3$

6. $y = \frac{x}{x^2 + 1}$

$y = \frac{x^2}{x^2 + 3}$

8. $y = \frac{x^2 + 1}{x^2 - 4}$

$y = \frac{3x}{x^2 - 1}$

10. $f(x) = \frac{x - 3}{x}$

11. $g(x) = x - \frac{8}{x^2}$

12. $f(x) = x + \frac{32}{x^2}$

13. $f(x) = \frac{x^2 + 1}{x}$

14. $f(x) = \frac{x^3}{x^2 - 9}$

15. $y = \frac{x^2 - 6x + 12}{x - 4}$

16. $y = \frac{2x^2 - 5x + 5}{x - 2}$

17. $y = x\sqrt{4 - x}$

18. $g(x) = x\sqrt{9 - x}$

19. $h(x) = x\sqrt{4 - x^2}$

20. $g(x) = x\sqrt{9 - x^2}$

21. $y = 3x^{2/3} - 2x$

22. $y = 3(x - 1)^{2/3} - (x - 1)^2$

23. $y = x^3 - 3x^2 + 3$

24. $y = -\frac{1}{3}(x^3 - 3x + 2)$

25. $y = 2 - x - x^3$

26. $f(x) = \frac{1}{3}(x - 1)^3 + 2$

27. $y = 3x^4 + 4x^3$

28. $y = 3x^4 - 6x^2 + \frac{5}{3}$

29. $y = x^5 - 5x$

30. $y = (x - 1)^5$

31. $y = |2x - 3|$

32. $y = |x^2 - 6x + 5|$

CAS **In Exercises 33–36, use a computer algebra system to analyze and graph the function. Identify any relative extrema, points of inflection, and asymptotes.**

33. $f(x) = \frac{20x}{x^2 + 1} - \frac{1}{x}$

34. $f(x) = x + \frac{4}{x^2 + 1}$

35. $f(x) = \frac{-2x}{\sqrt{x^2 + 7}}$

36. $f(x) = \frac{4x}{\sqrt{x^2 + 15}}$

In Exercises 37–46, sketch a graph of the function over the given interval. Use a graphing utility to verify your graph.

37. $f(x) = 2x - 4\sin x,\quad 0 \le x \le 2\pi$

38. $f(x) = -x + 2\cos x,\quad 0 \le x \le 2\pi$

39. $y = \sin x - \frac{1}{18}\sin 3x,\quad 0 \le x \le 2\pi$

40. $y = \cos x - \frac{1}{4}\cos 2x,\quad 0 \le x \le 2\pi$

41. $y = 2x - \tan x,\quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

42. $y = 2(x - 2) + \cot x,\quad 0 < x < \pi$

43. $y = 2(\csc x + \sec x),\quad 0 < x < \frac{\pi}{2}$

44. $y = \sec^2\left(\frac{\pi x}{8}\right) - 2\tan\left(\frac{\pi x}{8}\right) - 1,\quad -3 < x < 3$

45. $g(x) = x\tan x,\quad -\frac{3\pi}{2} < x < \frac{3\pi}{2}$

46. $g(x) = x\cot x,\quad -2\pi < x < 2\pi$

WRITING ABOUT CONCEPTS

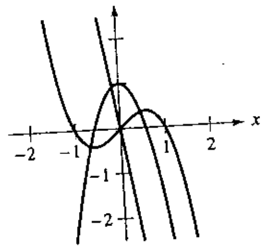
47. Suppose $f'(t) < 0$ for all t in the interval $(2, 8)$. Explain why $f(3) > f(5)$.

48. Suppose $f(0) = 3$ and $2 \le f'(x) \le 4$ for all x in the interval $[-5, 5]$. Determine the greatest and least possible values of $f(2)$.

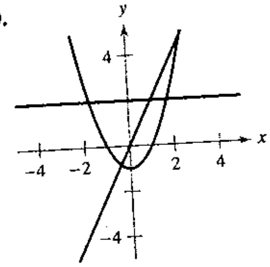
WRITING ABOUT CONCEPTS (continued)

In Exercises 49 and 50, the graphs of f, f', and f'' are shown on the same set of coordinate axes. Which is which? Explain your reasoning. To print an enlarged copy of the graph, go to the website ***www.mathgraphs.com.***

49.



50.



In Exercises 51–54, use a graphing utility to graph the function. Use the graph to determine whether it is possible for the graph of a function to cross its horizontal asymptote. Do you think it is possible for the graph of a function to cross its vertical asymptote? Why or why not?

51. $f(x) = \dfrac{4(x-1)^2}{x^2 - 4x + 5}$

52. $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

53. $h(x) = \dfrac{\sin 2x}{x}$

54. $f(x) = \dfrac{\cos 3x}{4x}$

In Exercises 55 and 56, use a graphing utility to graph the function. Explain why there is no vertical asymptote when a superficial examination of the function may indicate that there should be one.

55. $h(x) = \dfrac{6 - 2x}{3 - x}$

56. $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

In Exercises 57–60, use a graphing utility to graph the function and determine the slant asymptote of the graph. Zoom out repeatedly and describe how the graph on the display appears to change. Why does this occur?

57. $f(x) = -\dfrac{x^2 - 3x - 1}{x - 2}$

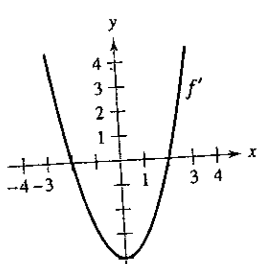
58. $g(x) = \dfrac{2x^2 - 8x - 15}{x - 5}$

59. $f(x) = \dfrac{2x^3}{x^2 + 1}$

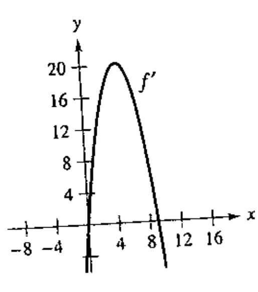
60. $h(x) = \dfrac{-x^3 + x^2 + 4}{x^2}$

Graphical Reasoning **In Exercises 61–64, use the graph of f' to sketch a graph of f and the graph of f''. To print an enlarged copy of the graph, go to the website** ***www.mathgraphs.com.***

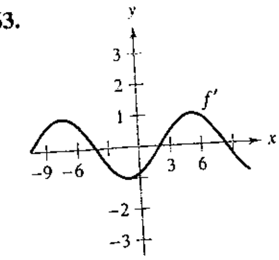
61.



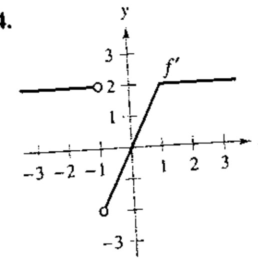
62.



63.



64.



(Submitted by Bill Fox, Moberly Area Community College, Moberly, MO)

CAS **65.** *Graphical Reasoning* Consider the function

$$f(x) = \frac{\cos^2 \pi x}{\sqrt{x^2 + 1}}, \quad 0 < x < 4.$$

(a) Use a computer algebra system to graph the function and use the graph to approximate the critical numbers visually.

(b) Use a computer algebra system to find f' and approximate the critical numbers. Are the results the same as the visual approximation in part (a)? Explain.

66. *Graphical Reasoning* Consider the function

$$f(x) = \tan(\sin \pi x).$$

(a) Use a graphing utility to graph the function.

(b) Identify any symmetry of the graph.

(c) Is the function periodic? If so, what is the period?

(d) Identify any extrema on $(-1, 1)$.

(e) Use a graphing utility to determine the concavity of the graph on $(0, 1)$.

Think About It **In Exercises 67–70, create a function whose graph has the given characteristics. (There is more than one correct answer.)**

67. Vertical asymptote: $x = 3$
Horizontal asymptote: $y = 0$

68. Vertical asymptote: $x = -5$
Horizontal asymptote: None

69. Vertical asymptote: $x = 3$
Slant asymptote: $y = 3x + 2$

70. Vertical asymptote: $x = 2$
Slant asymptote: $y = -x$

71. *Graphical Reasoning* The graph of f is shown in the figure on the next page.

(a) For which values of x is $f'(x)$ zero? Positive? Negative?

(b) For which values of x is $f''(x)$ zero? Positive? Negative?

(c) On what interval is f' an increasing function?

(d) For which value of x is $f'(x)$ minimum? For this value of x how does the rate of change of f compare with the rates of change of f for other values of x? Explain.

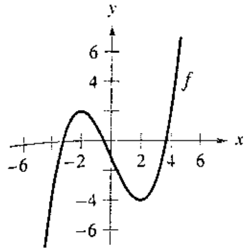


Figure for 71

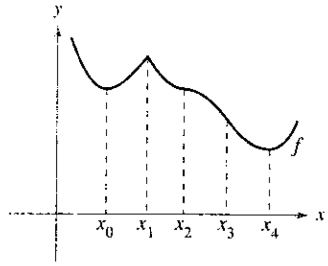


Figure for 72

CAPSTONE

72. ***Graphical Reasoning*** Identify the real numbers x_0, x_1, x_2, x_3, and x_4 in the figure such that each of the following is true.

(a) $f'(x) = 0$ (b) $f''(x) = 0$

(c) $f'(x)$ does not exist. (d) f has a relative maximum.

(e) f has a point of inflection.

73. ***Graphical Reasoning*** Consider the function

$$f(x) = \frac{ax}{(x-b)^2}.$$

Determine the effect on the graph of f as a and b are changed. Consider cases where a and b are both positive or both negative, and cases where a and b have opposite signs.

74. Consider the function $f(x) = \frac{1}{2}(ax)^2 - ax, a \neq 0$.

(a) Determine the changes (if any) in the intercepts, extrema, and concavity of the graph of f when a is varied.

(b) In the same viewing window, use a graphing utility to graph the function for four different values of a.

75. ***Investigation*** Consider the function

$$f(x) = \frac{2x^n}{x^4 + 1}$$

for nonnegative integer values of n.

(a) Discuss the relationship between the value of n and the symmetry of the graph.

(b) For which values of n will the x-axis be the horizontal asymptote?

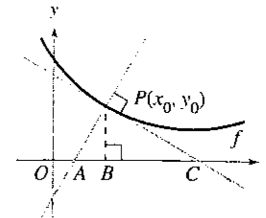
(c) For which value of n will $y = 2$ be the horizontal asymptote?

(d) What is the asymptote of the graph when $n = 5$?

(e) Use a graphing utility to graph f for the indicated values of n in the table. Use the graph to determine the number of extrema M and the number of inflection points N of the graph.

n	0	1	2	3	4	5
M						
N						

76. ***Investigation*** Let $P(x_0, y_0)$ be an arbitrary point on the graph of f such that $f'(x_0) \neq 0$, as shown in the figure. Verify each statement.



(a) The x-intercept of the tangent line is $\left(x_0 - \dfrac{f(x_0)}{f'(x_0)}, 0\right)$.

(b) The y-intercept of the tangent line is $(0, f(x_0) - x_0 f'(x_0))$.

(c) The x-intercept of the normal line is $(x_0 + f(x_0)f'(x_0), 0)$.

(d) The y-intercept of the normal line is $\left(0, y_0 + \dfrac{x_0}{f'(x_0)}\right)$.

(e) $|BC| = \left|\dfrac{f(x_0)}{f'(x_0)}\right|$ (f) $|PC| = \left|\dfrac{f(x_0)\sqrt{1 + [f'(x_0)]^2}}{f'(x_0)}\right|$

(g) $|AB| = |f(x_0)f'(x_0)|$

(h) $|AP| = |f(x_0)|\sqrt{1 + [f'(x_0)]^2}$

77. ***Modeling Data*** The data in the table show the number N of bacteria in a culture at time t, where t is measured in days.

	1	2	3	4	5	6	7	8
	25	200	804	1756	2296	2434	2467	2473

A model for these data is given by

$$N = \frac{24{,}670 - 35{,}153t + 13{,}250t^2}{100 - 39t + 7t^2}, \quad 1 \leq t \leq 8.$$

(a) Use a graphing utility to plot the data and graph the model.

(b) Use the model to estimate the number of bacteria when $t = 10$.

(c) Approximate the day when the number of bacteria is greatest.

CAS (d) Use a computer algebra system to determine the time when the rate of increase in the number of bacteria is greatest.

(e) Find $\lim_{t \to \infty} N(t)$.

Slant Asymptotes **In Exercises 78 and 79, the graph of the function has two slant asymptotes. Identify each slant asymptote. Then graph the function and its asymptotes.**

78. $y = \sqrt{4 + 16x^2}$ **79.** $y = \sqrt{x^2 + 6x}$

PUTNAM EXAM CHALLENGE

80. Let $f(x)$ be defined for $a \leq x \leq b$. Assuming appropriate properties of continuity and derivability, prove for $a < x < b$ that

$$\frac{\dfrac{f(x) - f(a)}{x - a} - \dfrac{f(b) - f(a)}{b - a}}{x - b} = \frac{1}{2}f''(\beta)$$

where β is some number between a and b.

3.7 Exercises

1. ***Numerical, Graphical, and Analytic Analysis*** Find two positive numbers whose sum is 110 and whose product is a maximum.

   (a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

First Number x	Second Number	Product P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$

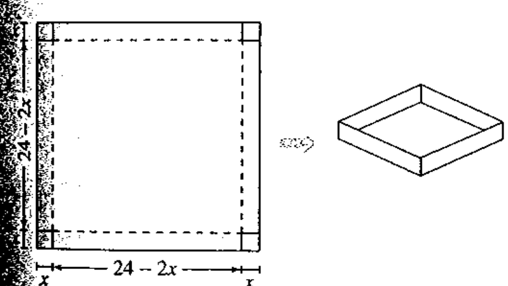
   (b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the solution. (*Hint:* Use the *table* feature of the graphing utility.)

   (c) Write the product P as a function of x.

   (d) Use a graphing utility to graph the function in part (c) and estimate the solution from the graph.

   (e) Use calculus to find the critical number of the function in part (c). Then find the two numbers.

2. ***Numerical, Graphical, and Analytic Analysis*** An open box of maximum volume is to be made from a square piece of material, 24 inches on a side, by cutting equal squares from the corners and turning up the sides (see figure).



   (a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.) Use the table to guess the maximum volume.

Height x	Length and Width	Volume V
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$

   (b) Write the volume V as a function of x.

   (c) Use calculus to find the critical number of the function in part (b) and find the maximum value.

   (d) Use a graphing utility to graph the function in part (b) and verify the maximum volume from the graph.

In Exercises 3–8, find two positive numbers that satisfy the given requirements.

3. The sum is S and the product is a maximum.
4. The product is 185 and the sum is a minimum.
5. The product is 147 and the sum of the first number plus three times the second number is a minimum.
6. The second number is the reciprocal of the first number and the sum is a minimum.
7. The sum of the first number and twice the second number is 108 and the product is a maximum.
8. The sum of the first number squared and the second number is 54 and the product is a maximum.

In Exercises 9 and 10, find the length and width of a rectangle that has the given perimeter and a maximum area.

9. Perimeter: 80 meters
10. Perimeter: P units

In Exercises 11 and 12, find the length and width of a rectangle that has the given area and a minimum perimeter.

11. Area: 32 square feet
12. Area: A square centimeters

In Exercises 13–16, find the point on the graph of the function that is closest to the given point.

	Function	*Point*		*Function*	*Point*
13.	$f(x) = x^2$	$\left(2, \frac{1}{2}\right)$	14.	$f(x) = (x - 1)^2$	$(-5, 3)$
15.	$f(x) = \sqrt{x}$	$(4, 0)$	16.	$f(x) = \sqrt{x - 8}$	$(12, 0)$

17. ***Area*** A rectangular page is to contain 30 square inches of print. The margins on each side are 1 inch. Find the dimensions of the page such that the least amount of paper is used.

18. ***Area*** A rectangular page is to contain 36 square inches of print. The margins on each side are $1\frac{1}{2}$ inches. Find the dimensions of the page such that the least amount of paper is used.

19. ***Chemical Reaction*** In an autocatalytic chemical reaction, the product formed is a catalyst for the reaction. If Q_0 is the amount of the original substance and x is the amount of catalyst formed, the rate of chemical reaction is

$$\frac{dQ}{dx} = kx(Q_0 - x).$$

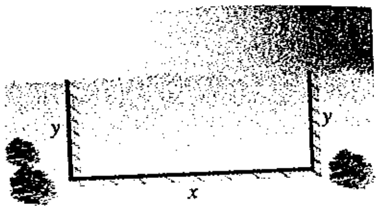
For what value of x will the rate of chemical reaction be greatest?

20. ***Traffic Control*** On a given day, the flow rate F (cars per hour) on a congested roadway is

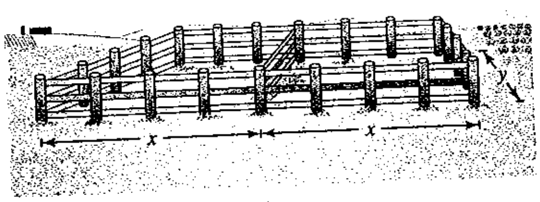
$$F = \frac{v}{22 + 0.02v^2}$$

where v is the speed of the traffic in miles per hour. What speed will maximize the flow rate on the road?

21. ***Area*** A farmer plans to fence a rectangular pasture adjacent to a river (see figure). The pasture must contain 245,000 square meters in order to provide enough grass for the herd. What dimensions will require the least amount of fencing if no fencing is needed along the river?



22. ***Maximum Area*** A rancher has 400 feet of fencing with which to enclose two adjacent rectangular corrals (see figure). What dimensions should be used so that the enclosed area will be a maximum?

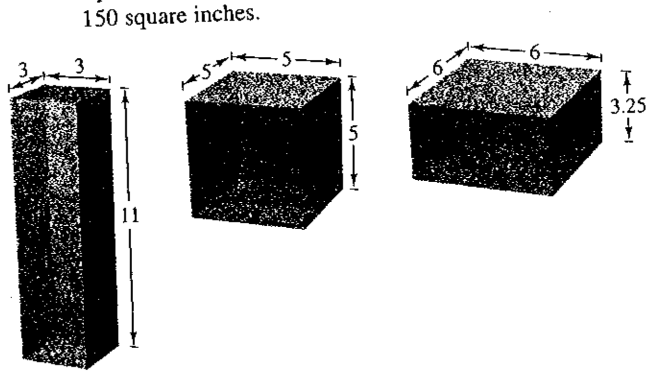


23. ***Maximum Volume***

(a) Verify that each of the rectangular solids shown in the figure has a surface area of 150 square inches.

(b) Find the volume of each solid.

(c) Determine the dimensions of a rectangular solid (with a square base) of maximum volume if its surface area is 150 square inches.



24. ***Maximum Volume*** Determine the dimensions of a rectangular solid (with a square base) with maximum volume if its surface area is 337.5 square centimeters.

25. ***Maximum Area*** A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window (see figure). Find the dimensions of a Norman window of maximum area if the total perimeter is 16 feet.

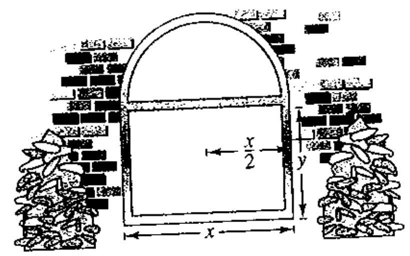


Figure for 25

26. ***Maximum Area*** A rectangle is bounded by the x- and y-axes and the graph of $y = (6 - x)/2$ (see figure). What length and width should the rectangle have so that its area is a maximum?

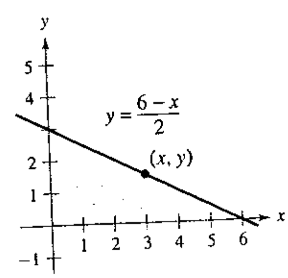


Figure for 26

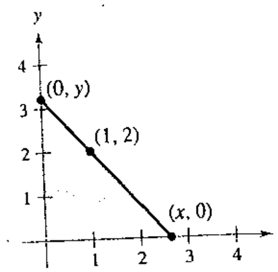


Figure for 27

27. ***Minimum Length*** A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(1, 2)$ (see figure).

(a) Write the length L of the hypotenuse as a function of x.

(b) Use a graphing utility to approximate x graphically such that the length of the hypotenuse is a minimum.

(c) Find the vertices of the triangle such that its area is a minimum.

28. ***Maximum Area*** Find the area of the largest isosceles triangle that can be inscribed in a circle of radius 6 (see figure).

(a) Solve by writing the area as a function of h.

(b) Solve by writing the area as a function of α.

(c) Identify the type of triangle of maximum area.

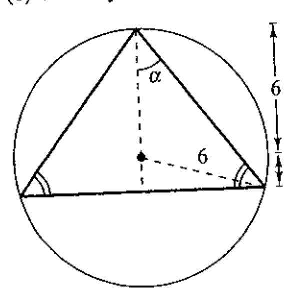


Figure for 28

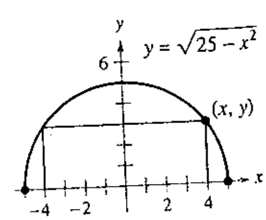


Figure for 29

29. ***Maximum Area*** A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$ (see figure). What length and width should the rectangle have so that its area is a maximum?

30. ***Area*** Find the dimensions of the largest rectangle that can be inscribed in a semicircle of radius r (see Exercise 29).

Numerical, Graphical, and Analytic Analysis An exercise room consists of a rectangle with a semicircle on each end. A 200-meter running track runs around the outside of the room.

(a) Draw a figure to represent the problem. Let x and y represent the length and width of the rectangle.

(b) Analytically complete six rows of a table such as the one below. (The first two rows are shown.) Use the table to guess the maximum area of the rectangular region.

Length x	Width y	Area xy
10	$\frac{2}{\pi}(100 - 10)$	$(10)\frac{2}{\pi}(100 - 10) \approx 573$
20	$\frac{2}{\pi}(100 - 20)$	$(20)\frac{2}{\pi}(100 - 20) \approx 1019$

(c) Write the area A as a function of x.

(d) Use calculus to find the critical number of the function in part (c) and find the maximum value.

(e) Use a graphing utility to graph the function in part (c) and verify the maximum area from the graph.

Numerical, Graphical, and Analytic Analysis A right circular cylinder is to be designed to hold 22 cubic inches of a soft drink (approximately 12 fluid ounces).

(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

Radius r	Height	Surface Area S
0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$

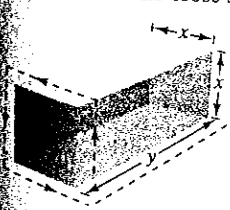
(b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the minimum surface area. (*Hint:* Use the *table* feature of the graphing utility.)

(c) Write the surface area S as a function of r.

(d) Use a graphing utility to graph the function in part (c) and estimate the minimum surface area from the graph.

(e) Use calculus to find the critical number of the function in part (c) and find dimensions that will yield the minimum surface area.

Maximum Volume A rectangular package to be sent by a postal service can have a maximum combined length and girth (perimeter of a cross section) of 108 inches (see figure). Find the dimensions of the package of maximum volume that can be sent. (Assume the cross section is square.)



34. ***Maximum Volume*** Rework Exercise 33 for a cylindrical package. (The cross section is circular.)

35. ***Maximum Volume*** Find the volume of the largest right circular cone that can be inscribed in a sphere of radius r.

36. ***Maximum Volume*** Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius r.

WRITING ABOUT CONCEPTS

37. A shampoo bottle is a right circular cylinder. Because the surface area of the bottle does not change when it is squeezed, is it true that the volume remains the same? Explain.

CAPSTONE

38. The perimeter of a rectangle is 20 feet. Of all possible dimensions, the maximum area is 25 square feet when its length and width are both 5 feet. Are there dimensions that yield a minimum area? Explain.

39. ***Minimum Surface Area*** A solid is formed by adjoining two hemispheres to the ends of a right circular cylinder. The total volume of the solid is 14 cubic centimeters. Find the radius of the cylinder that produces the minimum surface area.

40. ***Minimum Cost*** An industrial tank of the shape described in Exercise 39 must have a volume of 4000 cubic feet. The hemispherical ends cost twice as much per square foot of surface area as the sides. Find the dimensions that will minimize cost.

41. ***Minimum Area*** The sum of the perimeters of an equilateral triangle and a square is 10. Find the dimensions of the triangle and the square that produce a minimum total area.

42. ***Maximum Area*** Twenty feet of wire is to be used to form two figures. In each of the following cases, how much wire should be used for each figure so that the total enclosed area is maximum?

(a) Equilateral triangle and square

(b) Square and regular pentagon

(c) Regular pentagon and regular hexagon

(d) Regular hexagon and circle

What can you conclude from this pattern? {*Hint:* The area of a regular polygon with n sides of length x is $A = (n/4)[\cot(\pi/n)]x^2$.}

43. ***Beam Strength*** A wooden beam has a rectangular cross section of height h and width w (see figure on the next page). The strength S of the beam is directly proportional to the width and the square of the height. What are the dimensions of the strongest beam that can be cut from a round log of diameter 20 inches? (*Hint:* $S = kh^2w$, where k is the proportionality constant.)

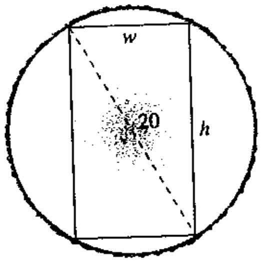


Figure for 43

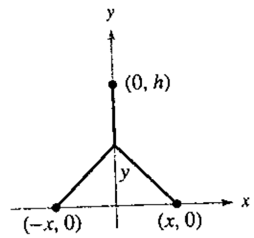


Figure for 44

44. *Minimum Length* Two factories are located at the coordinates $(-x, 0)$ and $(x, 0)$, and their power supply is at $(0, h)$ (see figure). Find y such that the total length of power line from the power supply to the factories is a minimum.

45. *Projectile Range* The range R of a projectile fired with an initial velocity v_0 at an angle θ with the horizontal is $R = \dfrac{v_0^2 \sin 2\theta}{g}$, where g is the acceleration due to gravity. Find the angle θ such that the range is a maximum.

46. *Conjecture* Consider the functions $f(x) = \frac{1}{2}x^2$ and $g(x) = \frac{1}{16}x^4 - \frac{1}{2}x^2$ on the domain $[0, 4]$.

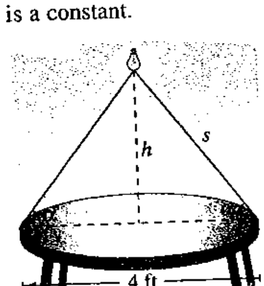
(a) Use a graphing utility to graph the functions on the specified domain.

(b) Write the vertical distance d between the functions as a function of x and use calculus to find the value of x for which d is maximum.

(c) Find the equations of the tangent lines to the graphs of f and g at the critical number found in part (b). Graph the tangent lines. What is the relationship between the lines?

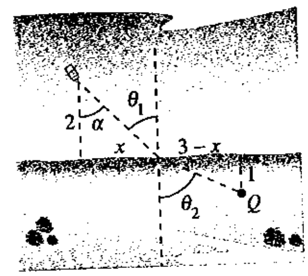
(d) Make a conjecture about the relationship between tangent lines to the graphs of two functions at the value of x at which the vertical distance between the functions is greatest, and prove your conjecture.

47. *Illumination* A light source is located over the center of a circular table of diameter 4 feet (see figure). Find the height h of the light source such that the illumination I at the perimeter of the table is maximum if $I = k(\sin \alpha)/s^2$, where s is the slant height, α is the angle at which the light strikes the table, and k is a constant.



48. *Illumination* The illumination from a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. Two light sources of intensities I_1 and I_2 are d units apart. What point on the line segment joining the two sources has the least illumination?

49. *Minimum Time* A man is in a boat 2 miles from the nearest point on the coast. He is to go to a point Q, located 3 miles down the coast and 1 mile inland (see figure). He can row at 2 miles per hour and walk at 4 miles per hour. Toward what point on the coast should he row in order to reach point Q in the least time?



50. *Minimum Time* Consider Exercise 49 if the point Q is on the shoreline rather than 1 mile inland.

(a) Write the travel time T as a function of α.

(b) Use the result of part (a) to find the minimum time to reach Q.

(c) The man can row at v_1 miles per hour and walk at v_2 miles per hour. Write the time T as a function of α. Show that the critical number of T depends only on v_1 and v_2 and not on the distances. Explain how this result would be more beneficial to the man than the result of Exercise 49.

(d) Describe how to apply the result of part (c) to minimizing the cost of constructing a power transmission cable that costs c_1 dollars per mile under water and c_2 dollars per mile over land.

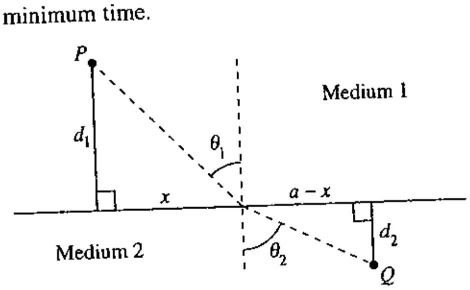
51. *Minimum Time* The conditions are the same as in Exercise 49 except that the man can row at v_1 miles per hour and walk at v_2 miles per hour. If θ_1 and θ_2 are the magnitudes of the angles, show that the man will reach point Q in the least time when

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.$$

52. *Minimum Time* When light waves traveling in a transparent medium strike the surface of a second transparent medium, they change direction. This change of direction is called *refraction* and is defined by **Snell's Law of Refraction,**

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$$

where θ_1 and θ_2 are the magnitudes of the angles shown in the figure and v_1 and v_2 are the velocities of light in the two media. Show that this problem is equivalent to that in Exercise 51, and that light waves traveling from P to Q follow the path of minimum time.



Sketch the graph of $f(x) = 2 - 2\sin x$ on the interval $[0, \pi/2]$.

(a) Find the distance from the origin to the y-intercept and the distance from the origin to the x-intercept.

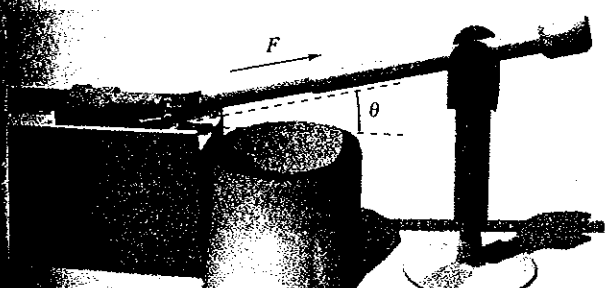
(b) Write the distance d from the origin to a point on the graph of f as a function of x. Use your graphing utility to graph d and find the minimum distance.

(c) Use calculus and the *zero* or *root* feature of a graphing utility to find the value of x that minimizes the function d on the interval $[0, \pi/2]$. What is the minimum distance?

(Submitted by Tim Chapell, Penn Valley Community College, Kansas City, MO)

Minimum Cost An offshore oil well is 2 kilometers off the coast. The refinery is 4 kilometers down the coast. Laying pipe in the ocean is twice as expensive as on land. What path should the pipe follow in order to minimize the cost?

Minimum Force A component is designed to slide a block of steel with weight W across a table and into a chute (see figure). The motion of the block is resisted by a frictional force proportional to its apparent weight. (Let k be the constant of proportionality.) Find the minimum force F needed to slide the block, and find the corresponding value of θ. (*Hint:* $F\cos\theta$ is the force in the direction of motion, and $F\sin\theta$ is the amount of force tending to lift the block. So, the apparent weight of the block is $W - F\sin\theta$.)



Maximum Volume A sector with central angle θ is cut from a circle of radius 12 inches (see figure), and the edges of the sector are brought together to form a cone. Find the magnitude of θ such that the volume of the cone is a maximum.

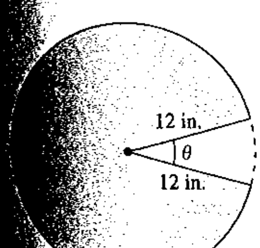


Figure for 56

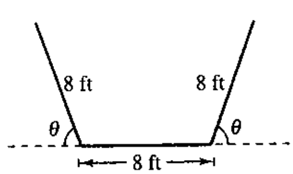


Figure for 57

57. ***Numerical, Graphical, and Analytic Analysis*** The cross sections of an irrigation canal are isosceles trapezoids of which three sides are 8 feet long (see figure). Determine the angle of elevation θ of the sides such that the area of the cross sections is a maximum by completing the following.

(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

Base 1	Base 2	Altitude	Area
8	$8 + 16\cos 10°$	$8\sin 10°$	≈ 22.1
8	$8 + 16\cos 20°$	$8\sin 20°$	≈ 42.5

(b) Use a graphing utility to generate additional rows of the table and estimate the maximum cross-sectional area. (*Hint:* Use the *table* feature of the graphing utility.)

(c) Write the cross-sectional area A as a function of θ.

(d) Use calculus to find the critical number of the function in part (c) and find the angle that will yield the maximum cross-sectional area.

(e) Use a graphing utility to graph the function in part (c) and verify the maximum cross-sectional area.

58. ***Maximum Profit*** Assume that the amount of money deposited in a bank is proportional to the square of the interest rate the bank pays on this money. Furthermore, the bank can reinvest this money at 12%. Find the interest rate the bank should pay to maximize profit. (Use the simple interest formula.)

59. ***Minimum Cost*** The ordering and transportation cost C of the components used in manufacturing a product is

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x+30}\right), \quad x \geq 1$$

where C is measured in thousands of dollars and x is the order size in hundreds. Find the order size that minimizes the cost. (*Hint:* Use the *root* feature of a graphing utility.)

60. ***Diminishing Returns*** The profit P (in thousands of dollars) for a company spending an amount s (in thousands of dollars) on advertising is

$$P = -\tfrac{1}{10}s^3 + 6s^2 + 400.$$

(a) Find the amount of money the company should spend on advertising in order to yield a maximum profit.

(b) The *point of diminishing returns* is the point at which the rate of growth of the profit function begins to decline. Find the point of diminishing returns.

Minimum Distance **In Exercises 61–63, consider a fuel distribution center located at the origin of the rectangular coordinate system (units in miles; see figures on next page). The center supplies three factories with coordinates (4, 1), (5, 6), and (10, 3). A trunk line will run from the distribution center along the line $y = mx$, and feeder lines will run to the three factories. The objective is to find m such that the lengths of the feeder lines are minimized.**

3.9 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find the equation of the tangent line T to the graph of f at the given point. Use this linear approximation to complete the table.

x	1.9	1.99	2	2.01	2.1
$f(x)$					
$T(x)$					

1. $f(x) = x^2$, $(2, 4)$
2. $f(x) = \dfrac{6}{x^2}$, $\left(2, \dfrac{3}{2}\right)$
3. $f(x) = x^5$, $(2, 32)$
4. $f(x) = \sqrt{x}$, $(2, \sqrt{2})$
5. $f(x) = \sin x$, $(2, \sin 2)$
6. $f(x) = \csc x$, $(2, \csc 2)$

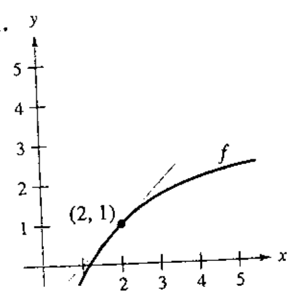
In Exercises 7–10, use the information to evaluate and compare Δy and dy.

7. $y = x^3$ $\quad x = 1 \quad \Delta x = dx = 0.1$
8. $y = 1 - 2x^2$ $\quad x = 0 \quad \Delta x = dx = -0.1$
9. $y = x^4 + 1$ $\quad x = -1 \quad \Delta x = dx = 0.01$
10. $y = 2 - x^4$ $\quad x = 2 \quad \Delta x = dx = 0.01$

In Exercises 11–20, find the differential dy of the given function.

11. $y = 3x^2 - 4$
12. $y = 3x^{2/3}$
13. $y = \dfrac{x + 1}{2x - 1}$
14. $y = \sqrt{9 - x^2}$
15. $y = x\sqrt{1 - x^2}$
16. $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$
17. $y = 3x - \sin^2 x$
18. $y = x \cos x$
19. $y = \dfrac{1}{3}\cos\left(\dfrac{6\pi x - 1}{2}\right)$
20. $y = \dfrac{\sec^2 x}{x^2 + 1}$

In Exercises 21–24, use differentials and the graph of f to approximate (a) $f(1.9)$ and (b) $f(2.04)$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

21.


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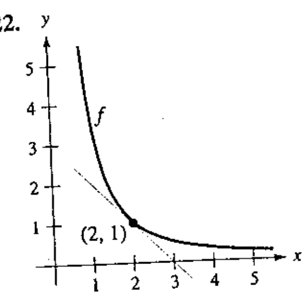

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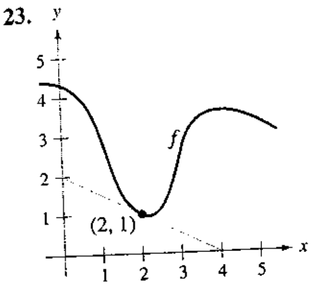

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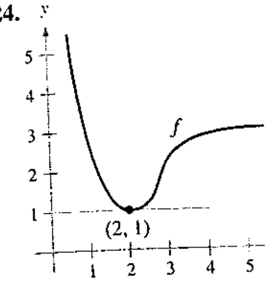
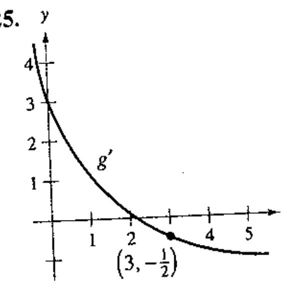


In Exercises 25 and 26, use differentials and the graph of g' to approximate (a) $g(2.93)$ and (b) $g(3.1)$ given that $g(3) = 8$.

25.


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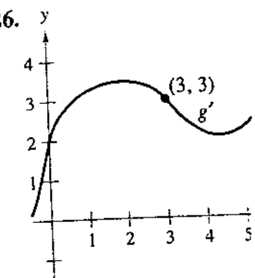


27. ***Area*** The measurement of the side of a square floor 10 inches, with a possible error of $\frac{1}{32}$ inch. Use different approximate the possible propagated error in computi area of the square.

28. ***Area*** The measurements of the base and altitude of a t are found to be 36 and 50 centimeters, respectivel possible error in each measurement is 0.25 centimete differentials to approximate the possible propagated e computing the area of the triangle.

29. ***Area*** The measurement of the radius of the end of a found to be 16 inches, with a possible error of $\frac{1}{4}$ in differentials to approximate the possible propagated computing the area of the end of the log.

30. ***Volume and Surface Area*** The measurement of the e cube is found to be 15 inches, with a possible error of 0. Use differentials to approximate the maximum propagated error in computing (a) the volume of the c (b) the surface area of the cube.

31. ***Area*** The measurement of a side of a square is fou 12 centimeters, with a possible error of 0.05 centimete
    (a) Approximate the percent error in computing the ar square.
    (b) Estimate the maximum allowable percent error in n the side if the error in computing the area cannc 2.5%.

32. ***Circumference*** The measurement of the circumfer circle is found to be 64 centimeters, with a possible 0.9 centimeter.
    (a) Approximate the percent error in computing the a circle.

(b) Estimate the maximum allowable percent error in measuring the circumference if the error in computing the area cannot exceed 3%.

33. ***Volume and Surface Area*** The radius of a spherical balloon is measured as 8 inches, with a possible error of 0.02 inch. Use differentials to approximate the maximum possible error in calculating (a) the volume of the sphere, (b) the surface area of the sphere, and (c) the relative errors in parts (a) and (b).

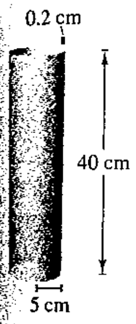
34. ***Stopping Distance*** The total stopping distance T of a vehicle is

$$T = 2.5x + 0.5x^2$$

where T is in feet and x is the speed in miles per hour. Approximate the change and percent change in total stopping distance as speed changes from $x = 25$ to $x = 26$ miles per hour.

Volume **In Exercises 35 and 36, the thickness of each shell is 0.2 centimeter. Use differentials to approximate the volume of each shell.**

35.



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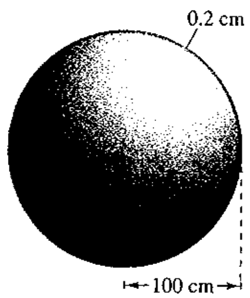

37. ***Pendulum*** The period of a pendulum is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length of the pendulum in feet, g is the acceleration due to gravity, and T is the time in seconds. The pendulum has been subjected to an increase in temperature such that the length has increased by $\frac{1}{2}\%$.

(a) Find the approximate percent change in the period.

(b) Using the result in part (a), find the approximate error in this pendulum clock in 1 day.

Ohm's Law A current of I amperes passes through a resistor of R ohms. **Ohm's Law** states that the voltage E applied to the resistor is $E = IR$. If the voltage is constant, show that the magnitude of the relative error in R caused by a change in I is equal in magnitude to the relative error in I.

Triangle Measurements The measurement of one side of a right triangle is found to be 9.5 inches, and the angle opposite that side is 26°45′ with a possible error of 15′.

(a) Approximate the percent error in computing the length of the hypotenuse.

(b) Estimate the maximum allowable percent error in measuring the angle if the error in computing the length of the hypotenuse cannot exceed 2%.

40. ***Area*** Approximate the percent error in computing the area of the triangle in Exercise 39.

41. ***Projectile Motion*** The range R of a projectile is

$$R = \frac{v_0^2}{32}(\sin 2\theta)$$

where v_0 is the initial velocity in feet per second and θ is the angle of elevation. If $v_0 = 2500$ feet per second and θ is changed from 10° to 11°, use differentials to approximate the change in the range.

42. ***Surveying*** A surveyor standing 50 feet from the base of a large tree measures the angle of elevation to the top of the tree as 71.5°. How accurately must the angle be measured if the percent error in estimating the height of the tree is to be less than 6%?

In Exercises 43–46, use differentials to approximate the value of the expression. Compare your answer with that of a calculator.

43. $\sqrt{99.4}$ **44.** $\sqrt[3]{26}$

45. $\sqrt[4]{624}$ **46.** $(2.99)^3$

In Exercises 47 and 48, verify the tangent line approximation of the function at the given point. Then use a graphing utility to graph the function and its approximation in the same viewing window.

	Function	*Approximation*	*Point*
47.	$f(x) = \sqrt{x + 4}$	$y = 2 + \frac{x}{4}$	$(0, 2)$
48.	$f(x) = \tan x$	$y = x$	$(0, 0)$

WRITING ABOUT CONCEPTS

49. Describe the change in accuracy of dy as an approximation for Δy when Δx is decreased.

50. When using differentials, what is meant by the terms *propagated error*, *relative error*, and *percent error*?

51. Give a short explanation of why the approximation is valid.

(a) $\sqrt{4.02} \approx 2 + \frac{1}{4}(0.02)$

(b) $\tan 0.05 \approx 0 + 1(0.05)$

CAPSTONE

52. Would you use $y = x$ to approximate $f(x) = \sin x$ near $x = 0$? Why or why not?

True or False? **In Exercises 53–56, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

53. If $y = x + c$, then $dy = dx$.

54. If $y = ax + b$, then $\Delta y/\Delta x = dy/dx$.

55. If y is differentiable, then $\lim_{\Delta x \to 0} (\Delta y - dy) = 0$.

56. If $y = f(x)$, f is increasing and differentiable, and $\Delta x > 0$, then $\Delta y \geq dy$.