What is Mathematics?

A NAIVE DEFINITION, adequate for the dictionary and for an initial understanding, is that *mathematics is the science of quantity and space*. Expanding this definition a bit, one might add that mathematics also deals with the symbolism relating to quantity and to space.

This definition certainly has a historical basis and will serve us for a start, but it is one of the purposes of this work to modify and amplify it in a way that reflects the growth of the subject over the past several centuries and indicates the visions of various schools of mathematics as to what the subject ought to be.

The sciences of quantity and of space in their simpler forms are known as *arithmetic* and *geometry*. Arithmetic, as taught in grade school, is concerned with numbers of various sorts, and the rules for operations with numbers—addition, subtraction, and so forth. And it deals with situations in daily life where these operations are used.

Geometry is taught in the later grades. It is concerned in part with questions of spatial measurements. If I draw such a line and another such line, how far apart will their end points be? How many square inches are there in a rectangle 4 inches long and 8 inches wide? Geometry is also concerned with aspects of space that have a strong aesthetic appeal or a surprise element. For example, it tells us that in any parallelogram whatsoever, the diagonals bisect one another; in any triangle whatsoever, the three medians intersect in a common point. It teaches us that a floor can be tiled with equilateral triangles or hexagons, but not with regular pentagons.

But geometry, if taught according to the arrangement laid out by Euclid in 300 B.C., has another vitally significant aspect. This is its presentation as a deductive science. Beginning with a number of elementary ideas which are assumed to be self-evident, and on the basis of a few definite rules of mathematical and logical manipulation, Euclidean geometry builds up a fabric of deductions of increasing complexity.

What is stressed in the teaching of elementary geometry is not only the spatial or visual aspect of the subject but the methodology wherein hypothesis leads to conclusion. This deductive process is known as *proof*. Euclidean geometry is the first example of a formalized deductive system and has become the model for all such systems. Geometry has been the great practice field for logical thinking, and the study of geometry has been held (rightly or wrongly) to provide the student with a basic training in such thinking.

Although the deductive aspects of arithmetic were clear to ancient mathematicians, these were not stressed either in teaching or in the creation of new mathematics until the 1800s. Indeed, as late as the 1950s one heard statements from secondary school teachers, reeling under the impact of the "new math," to the effect that they had always thought geometry had "proof" while arithmetic and algebra did not.

With the increased emphasis placed on the deductive aspects of all branches of mathematics, C. S. Peirce in the middle of the nineteenth century, announced that "mathematics is the science of making necessary conclusions." Conclusions about what? About quantity? About space? The content of mathematics is not defined by this definition; mathematics could be "about" anything as long as it is a subject that exhibits the pattern of assumption-deduction-conclusion. Sherlock Holmes remarks to Watson in *The Sign of Four* that "Detection is, or ought to be, an exact science and should be treated in the same cold and unemotional manner. You have attempted to tinge it with romanticism, which produces much the same effect as if you worked a love-story or an elopement into the fifth proposition of Euclid." Here Conan Doyle, with tongue in cheek, is asserting that criminal detection might very well be considered a branch of mathematics. Peirce would agree.

The definition of mathematics changes. Each generation and each thoughtful mathematician within a generation formulates a definition according to his lights. We shall examine a number of alternate formulations before we write Finis to this volume.

Further Readings. See Bibliography

A. Alexandroff; A. Kolmogoroff and M. Lawrentieff; R. Courant and H. Robbins; T. Danzig [1959]; H. Eves and C. Newsom; M. Gaffney and L. Steen; N. Goodman; E. Kasner and J. Newman; R. Kershner and L. Wilcox; M. Kline [1972]; A. Kolmogoroff; J. Newman [1956]; E. Snapper; E. Stabler; L. Steen [1978]

Where is Mathematics?

WHERE IS THE PLACE of mathematics? Where does it exist? On the printed page, of course, and prior to printing, on tablets or on papyri. Here is a mathematical book—take it in your hand; you have a palpable record of mathematics as an intellectual endeavor. But first it must exist in people's minds, for a shelf of books doesn't create mathematics. Mathematics exists on taped lectures, in computer memories and printed circuits. Should we say also that it resides in mathematical machines such as slide rules and cash registers and, as some believe, in the arrangement of the stones at Stonehenge? Should we say that it resides in the genes of the sunflower plant if that plant brings forth seeds arranged in Bernoullian spirals and transmits mathematical information from generation to generation? Should we say that mathematics exists on a wall if a lampshade casts a parabolic shadow on that wall? Or do we believe that all these are mere shadow manifestations of the real mathematics which, as some philosophers have asserted, exists eternally and independently of this actualized universe, independently of all possible actualizations of a universe?

What is knowledge, mathematical or otherwise? In a correspondence with the writer, Sir Alfred Ayer suggests that one of the leading dreams of philosophy has been "to agree on a criterion for deciding what there is," to which we might add, "and for deciding where it is to be found."

The Mathematical Community

THERE IS HARDLY a culture, however primitive, which does not exhibit some rudimentary kind of mathematics. The mainstream of western mathematics as a systematic pursuit has its origin in Egypt and Mesopotamia. It spread to Greece and to the Graeco-Roman world. For some 500 years following the fall of Rome, the fire of mathematical creativeness was all but extinguished in Europe; it is thought to have been preserved in Persia. After some centuries of inactivity, the flame appeared again in the Islamic world and from there mathematical knowledge and enthusiasm spread through Sicily and Italy to the whole of Europe.

A rough timetable would be

Egyptian:	3000 B.C. to 1600 B.C.
Babylonian:	1700 B.C. to 300 B.C.
Greek:	600 B.C. to 200 B.C.
Graeco-Roman:	150 A.D. to 525 A.D.

Islamic:	750 A.D. to 1450 A.D.
Western:	1100 A.D. to 1600 A.D.
Modern:	1600 A.D. to present.

Other streams of mathematical activity are the Chinese, the Japanese, the Hindu, and the Inca-Aztec. The interaction between western and eastern mathematics is a subject of scholarly investigation and conjecture.

At the present time, there is hardly a country in the world which is not creating new mathematics. Even the emerging nations, so called, all wish to establish up-to-date university programs in mathematics, and the hallmark of excellence is taken to be the research activity of their staffs.

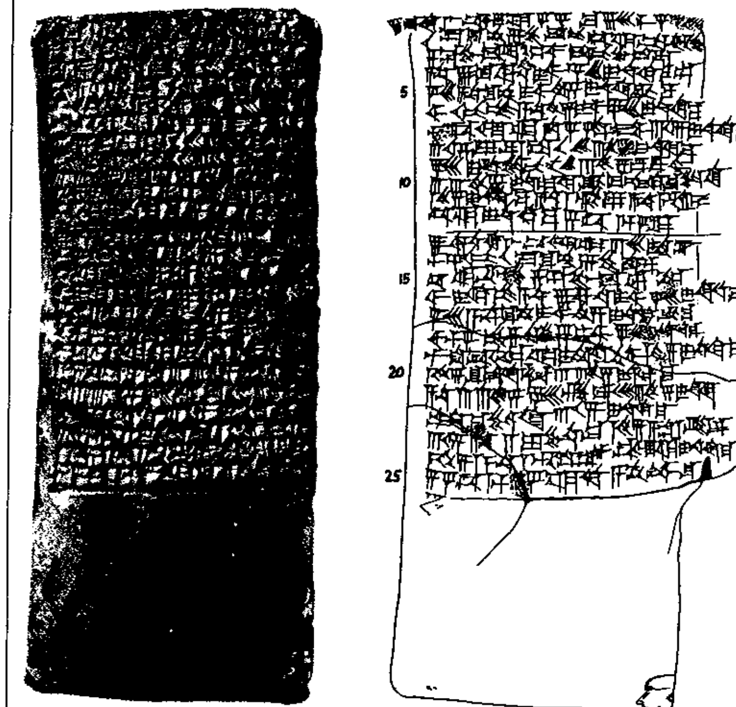
In contrast to the relative isolation of early oriental and western mathematics from each other, the mathematics of today is unified. It is worked and transmitted in full and open knowledge. Personal secrecy like that practiced by the Renaissance and Baroque mathematicians hardly exists. There is a vast international network of publications; there are national and international open meetings and exchanges of scholars and students.

François Viète
1540–1603

In all honesty, though, it should be admitted that restriction of information has occurred during wartime. There is also considerable literature on mathematical cryptography, as practiced by the professional cryptographers, which is not, for obvious reasons, generally available.

In the past mathematics has been pursued by people in various walks of life. Thomas Bradwardine (1325) was Archbishop of Canterbury. Ulugh Beg with his trigonometric tables was the grandson of Tamerlane. Luca Pacioli (1470) was a monk. Ferrari (1548) was a tax assessor. Cardano (1550) was professor of medicine. Viète (1580) was a lawyer in the royal privy council. Van Ceulen (1610) was a fencing master. Fermat (1635) was a lawyer. Many mathematicians earned part of their living as protegés of the Crown: John Dee, Kepler, Descartes, Euler; some even had the title of "Mathematicus." Up to about 1600, a mathematician could earn a few pounds by casting horoscopes or writing amulets for the wealthy.

René Descartes
1596–1650



Explanation: The drawing is a contemporary version of the symbols on the clay tablet. A line by line translation of the first twelve lines is given. The notation 3;3,45 used in the translation means $3 + \frac{3}{60} + \frac{45}{3600} = 3.0625$. In modern terms, the problem posed by this tablet is: given $x + y$ and xy, find x and y. Solution:

$$\begin{matrix} x = \\ y = \end{matrix} \frac{x+y}{2} \pm \sqrt{\left(\frac{x+y}{2}\right)^2 - xy}$$

What mathematics looked like in 1700 B.C. *Clay tablet with cuneiform writing from southern Iraq. The two problems that are worked out follow the standard procedure in Babylonian mathematics for quadratic equations.*

[1] 9 (gín) is the (total expenses in) silver of a kilá; I added the length and the width, and (the result is) 6;30 (GAR); ½ GAR is [its depth],
[2] 10 gin (volume) the assignment, 6 še (silver) the wages. What are the length (and) its width?
[3] When you perform (the operations), take the reciprocal of the wages,
[4] multiply by 9 gin, the (total expenses in) silver, (and) you will get 4,30;
[5] multiply 4,30 by the assignment, (and) you will get 45;
[6] take the reciprocal of its depth, multiply by 45, (and) you will get 7;30;
[7] halve the length and the width which I added together, (and) you will get 3;15;
[8] square 3;15, (and) you will get 10;33,45;
[9] subtract 7;30 from 10;33,45, (and)
[10] you will get 3;3,45; take its square root, (and)
[11] you will get 1;45; add it to the one, subtract it from the other, (and)
[12] you will get the length (and) the width. 5 (GAR) is the length; 1½ GAR is the width.

Courtesy: Prof. A. J. Sachs, from Neugebauer and Sachs, "Mathematical Cuneiform Texts"

These days there is nothing to prevent a wealthy person from pursuing mathematics full or part time in isolation, as in the era when science was an aristocrat's hobby. But this kind of activity is now not at a sufficiently high voltage to sustain invention of good quality. Nor does the church (or the monarchy) support mathematics as it once did.

For the past century, universities have been our principal sponsors. By releasing part of his time, the university encourages a lecturer to engage in mathematical research. At present, most mathematicians are supported directly or indirectly by the university, by corporations such as IBM, or by the federal government, which in 1977 spent about $130,000,000 on mathematics of all sorts.

To the extent that all children learn some mathematics, and that a certain small fraction of mathematics is in the common language, the mathematics community and the community at large are identical. At the higher levels of practice, at the levels where new mathematics is created and transmitted, we are a fairly small community. The combined membership list of the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics for the year 1978 lists about 30,000 names. It is by no means necessary for one to think of oneself as a mathematician to operate at the highest mathematical levels; one might be a physicist, an engineer, a computer scientist, an economist, a geographer, a statistician or a psychologist. Perhaps the American mathematical community should be reckoned at 60 or 90 thousand with corresponding numbers in all the developed or developing countries.

Numerous regional, national, and international meetings are held periodically. There is lively activity in the writing and publishing of books at all levels, and there are more than 1600 individual technical journals to which it is appropriate to submit mathematical material.

These activities make up an international forum in which mathematics is perpetuated and innovated; in which discrepancies in practice and meaning are thrashed out.

Further Readings. See Bibliography

R. Archibald; E. Bell; B. Boos and M. Niss; C. Boyer; F. Cajori; J. S. Frame; R. Gillings; E. Husserl; M. Kline [1972]; U. Libbrecht; Y. Mikami; J. Needham; O. Neugebauer; O. Neugebauer and A. Sachs; D. Struik; B. Van der Waerden

The Tools of the Trade

WHAT AUXILIARY TOOLS or equipment are necessary for the pursuit of mathematics? There is a famous picture showing Archimedes poring over a problem drawn in the sand while Roman soldiers lurk menacingly in the background. This picture has penetrated the psyche of the profession and has helped to shape its external image. It tells us that mathematics is done with a minimum of tools—a bit of sand, perhaps, and an awful lot of brains.

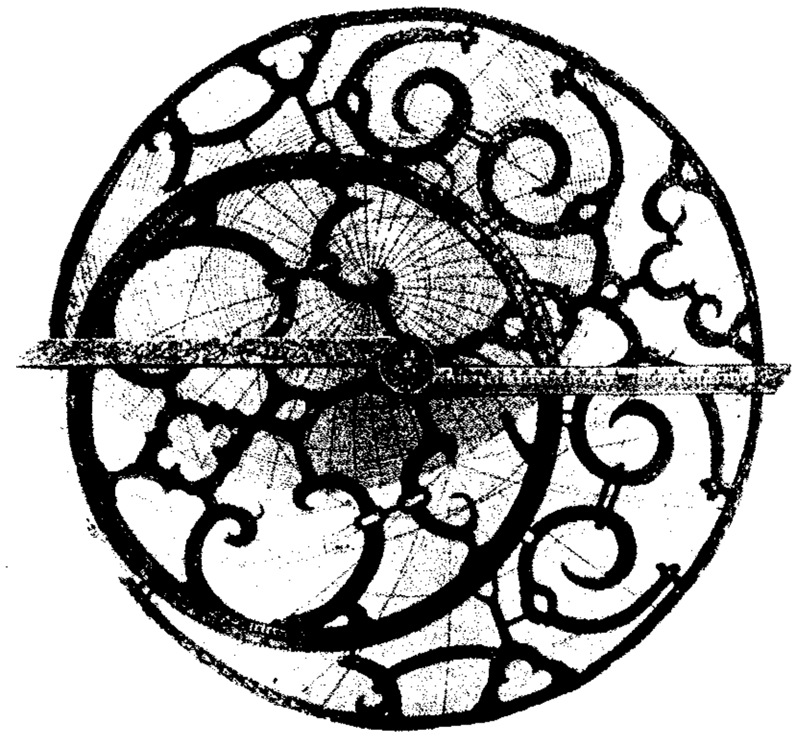
Some mathematicians like to think that it could even be done in a dark closet by a solitary man drawing on the resources of a brilliant platonic intellect. It is true that mathematics does not require vast amounts of laboratory equipment, that "Gedankenexperimente" (thought-experiments) are largely what is needed. But it is by no means fair to say that mathematics is done totally in the head.

Perhaps, in very ancient days, primitive mathematics, like the great epics and like ancient religions, was transmitted by oral tradition. But it soon became clear that to do mathematics one must have, at the very least, instruments of writing or recording and of duplication. Before the invention of printing, there were "scribe factories" for the wholesale replication of documents.

The ruler and compass are built into the axioms at the foundation of Euclidean geometry. Euclidean geometry can be defined as the science of ruler-and-compass constructions.

Arithmetic has been aided by many instruments and devices. Three of the most successful have been the abacus, the slide rule, and the modern electronic computer. And, the logical capabilities of the computer have already relegated its arithmetic skills to secondary importance.

In the beginning, we used to count computers. There were four: one in Philadelphia, one in Aberdeen, one in



Astrolabe, 1568.

Cambridge, and one in Washington. Then there were ten. Then, suddenly, there were two hundred. The last figure heard was thirty–five thousand. The computers proliferated, and generation followed generation, until now the fifty dollar hand-held job packs more computing power than the hippopotamian hulks rusting in the Smithsonian: the ENIACS, the MARKS, the SEACS, and the GOLEMS. Perhaps tomorrow the $1.98 computer will flood the drugstores and become a throwaway object like a plastic razor or a piece of Kleenex.

Legend has it that in the late 1940s when old Tom Watson of the IBM corporation learned of the potentialities of the computer he estimated that two or three of them would take care of the needs of the nation. Neither he nor anyone else foresaw how the mathematical needs of the nation would rise up miraculously to fill the available computing power.

The relationship of computers to mathematics has been far more complex than laymen might suspect. Most people assume that anyone who calls himself a professional mathematician uses computing machines. In truth, compared to engineers, physicists, chemists, and economists, most mathematicians have been indifferent to and ignorant of the use of computers. Indeed, the notion that creative mathematical work could ever be mechanized seems, to many mathematicians, demeaning to their professional self-esteem. Of course, to the applied mathematician, working along with scientists and engineers to get numerical answers to practical questions, the computer has been an indispensable assistant for many years.

When programmed appropriately, the computer also has the ability to perform many symbolic mathematical operations. For example, it can do formal algebra, formal calculus, formal power series expansions and formal work in differential equations. It has been thought that a program like FORMAC or MACSYMA would be an invaluable aid to the applied mathematician. But this has not yet been the case, for reasons which are not clear.

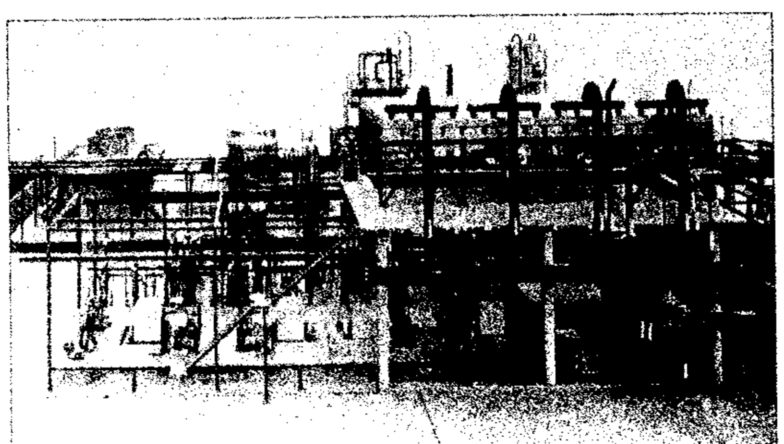
In geometry, the computer is a drawing instrument of much greater power than any of the linkages and templates of the traditional drafting room. Computer graphics show beautifully shaded and colored pictures of "objects" which are only mathematically or programatically defined. The viewer would swear that these images are projected photographs of real objects. But he would be wrong; the "objects" depicted have no "real world" existence. In some cases, they could not possibly have such existence.

On the other hand, it is still sometimes more efficient to use a physical model rather than attempting a computer graphics display. A chemical engineering firm, with whose practice the writer is familiar, designs plants for the petrochemical industry. These plants often have reticulated piping arrangements of a very complicated nature. It is standard company practice to build a scaled, color-coded model from little plastic Tinker Toy parts and to work in a significant way with this physical model.

The computer served to intensify the study of numerical analysis and to wake matrix theory from a fifty-year slumber. It called attention to the importance of logic and of the theory of discrete abstract structures. It led to the creation of new disciplines such as linear programming and the study of computational complexity.

Occasionally, as with the four-color problem (see Chapter 8.), it lent a substantial assist to a classical unsolved problem, as a helicopter might rescue a Conestoga wagon from sinking in the mud of the Pecos River. But all these effects were marginal. Most mathematical research continued to go on just as it would have if the computing machine did not exist.

Within the last few years, however, computers have had a noticeable impact in the field of pure mathematics. This may be the result of the arrival of a generation of mathematicians who learned computer programming in high school and to whom a computer terminal is as familiar as a telephone or a bicycle. One begins to see a change in mathematical research. There is greater interest in constructive and algorithmic results, and decreasing interest in purely existential or dialectical results that have little or no computational meaning. (See Chapter 4 for further discussion of these issues.) The fact that computers are available affects mathematics by luring mathematicians to move in directions where the computer can play a part. Nevertheless, it is true, even today, that most mathematical research is carried on without any actual or potential use of computers.



Plastic model used by engineering firm.

Courtesy: The Lummus Co., Bloomfield, N.J.

Further Readings. See Bibliography

D. Hartree; W. Meyer zur Capellen; F. J. Murray [1961]; G. R. Stibitz; M. L. Dertouzos and J. Moses; H. H. Goldstine [1972] [1977]; I. Taviss; P. Henrici [1974]; J. Traub

How Much Mathematics Is Now Known?

THE MATHEMATICS BOOKS at Brown University are housed on the fifth floor of the Sciences Library. In the trade, this is commonly regarded as a fine mathematical collection, and a rough calculation shows that this floor contains the equivalent of 60,000 average-sized volumes. Now there is a certain redundancy in the contents of these volumes and a certain deficiency in the Brown holdings, so let us say these balance out. To this figure we should add, perhaps, an equal quantity of mathematical material in adjacent areas such as engineering, physics, astronomy, cartography, or in new applied areas such as economics. In this way, we arrive at a total of, say, 100,000 volumes.

One hundred thousand volumes. This amount of knowledge and information is far beyond the comprehension of any one person. Yet it is small compared to other collections, such as physics, medicine, law, or literature. Within the lifetime of a man living today, the *whole* of mathematics was considered to be essentially within the grasp of a devoted student. The Russian-Swiss mathematician Alexanalysis and to wake matrix theory from a fifty-year slumber. It called attention to the importance of logic and of the theory of discrete abstract structures. It led to the creation of new disciplines such as linear programming and the study of computational complexity.

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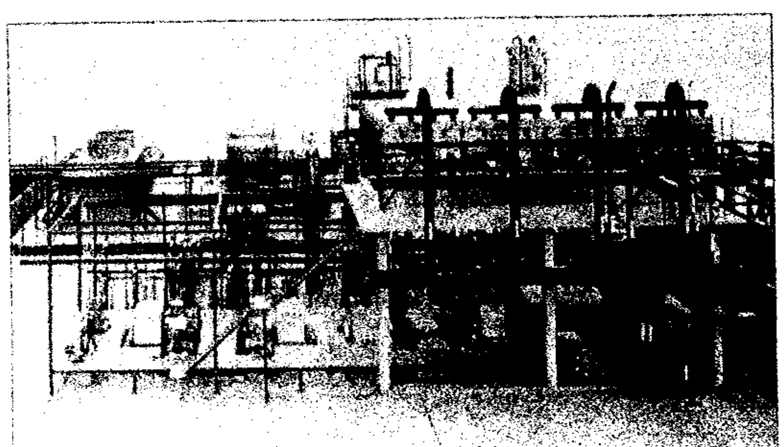

Plastic model used by engineering firm.

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John von Neumann
1903–1957

The same assertion would not be made today. In the late 1940s, John von Neumann estimated that a skilled mathematician might know, in essence, ten percent of what was available. There is a popular saying that knowledge always adds, never subtracts. This saying persists despite such shocking assessments as that of A. N. Whitehead who observed that Europe in 1500 knew less than Greece knew at the time of Archimedes. Mathematics builds on itself; it is aggregative. Algebra builds on arithmetic. Geometry builds on arithmetic and on algebra. Calculus builds on all three. Topology is an offshoot of geometry, set theory, and algebra. Differential equations builds on calculus, topology, and algebra. Mathematics is often depicted as a mighty tree with its roots, trunk, branches, and twigs labelled according to certain subdisciplines. It is a tree that grows in time.

Constructs are enlarged and filled in. New theories are created. New mathematical objects are delineated and put under the spotlight. New relations and interconnections are found, thereby expressing new unities. New applications are sought and devised.

As this occurs, what is old and true is retained—at least in principle. Everything that once was mathematics remains mathematics—at least in principle. And so it would appear that the subject is a vast, increasing organism, with branch upon branch of theory and practice. The prior branch is prerequisite for the understanding of the subsequent branch. Thus, the student knows that in order to study and understand the theory of differential equations, he should have had courses in elementary calculus and in linear algebra. This serial dependence is in contrast to other disciplines, such as art or music. One can like or "understand" modern art without being familiar with baroque art; one can create jazz without any grounding in seventeenth century madrigals.

But while there is much truth in the view of mathematics as a cumulative science, this view as presented is somewhat naive. As mathematical textures are built up, there are concomitantly other processes at work which tend to break them down. Individual facts are found to be erroneous or incomplete. Theories become unpopular and are neglected. Work passes into obscurity and becomes grist for the mill of antiquarians (as with, say, prosthaphaeresic multiplication*). Other theories become saturated and are not pursued further. Older work is seen from modern perspectives and is recast, reformulated, while the older formulations may even become unintelligible (Newton's original writings can now be interpreted only by specialists). Applications become irrelevant and forgotten (the aerodynamics of Zeppelins). Superior methods are discovered and replace inferior ones (vast tables of special functions for computation are replaced by the wired-in approximations of the digital computer). All this contributes to a diminution of the material that must be held in the forefront of the mathematical consciousness.

There is also a loss of knowledge due to destruction or deterioration of the physical record. Libraries have been destroyed in wars and in social upheavals. And what is not accomplished by wars may be done by chemistry. The paper used in the early days of printing was much finer than what is used today. Around 1850 cheap, wood-pulp paper with acid-forming coatings was introduced, and the self-destructive qualities of this combination, together with our polluted atmosphere, can lead to the crumbling of pages as a book is read.

How many mathematics books should the Ph.D. candidate in mathematics know? The average candidate will take about fourteen to eighteen semester courses of undergraduate mathematics and sixteen graduate courses. At one book per course, and then doubling the answer for collateral and exploratory reading, we arrive at a figure of

* I.e., multiplication carried out by the addition of trigonometric functions.

about sixty to eighty volumes. In other words, two shelves of books will do the trick. This is a figure well within the range of human comprehension; it has to be.

Thus we can think of our 60,000 books as an ocean of knowledge, with an average depth of sixty or seventy books. At different locations within this ocean—that is, at different subspecialties within mathematics—we can take a depth sample, the two-foot bookshelf that would represent the basic education of a specialist in that area. Dividing 60,000 books by sixty, we find there should be at least 1,000 distinct subspecialties. But this is an underestimate, for many books would appear on more than one subspecialty's basic booklist. The coarse subdivision of mathematics, according to the AMS (MOS) Classification Scheme of 1980, is given in Appendix *B*. The fine structure would show mathematical writing broken down into more than 3,000 categories.

In most of these 3,000 categories, new mathematics is being created at a constantly increasing rate. The ocean is expanding, both in depth and in breadth.

Further Readings. See Bibliography

J. von Neumann; C. S. Fisher

Ulam's Dilemma

WE CAN USE THE TERM "Ulam's dilemma" for the situation which Stanislaw Ulam has described vividly in his autobiography, *Adventures of a Mathematician.*

"At a talk which I gave at a celebration of the twenty-fifth anniversary of the construction of von Neumann's computer in Princeton a few years ago, I suddenly started estimating silently in my mind how many theorems are published yearly in mathematical journals. I made a quick

mental calculation and came to a number like one hundred thousand theorems per year. I mentioned this and my audience gasped. The next day two of the younger mathematicians in the audience came to tell me that, impressed by this enormous figure, they undertook a more systematic and detailed search in the Institute library. By multiplying the number of journals by the number of yearly issues, by the number of papers per issue and the average number of theorems per paper, their estimate came to nearly two hundred thousand theorems a year. If the number of theorems is larger than one can possibly survey, who can be trusted to judge what is 'important'? One cannot have survival of the fittest if there is no interaction. It is actually impossible to keep abreast of even the more outstanding and exciting results. How can one reconcile this with the view that mathematics will survive as a single science? In mathematics one becomes married to one's own little field. Because of this, the judgment of value in mathematical research is becoming more and more difficult, and most of us are becoming mainly technicians. The variety of objects worked on by young scientists is growing exponentially. Perhaps one should not call it a pollution of thought; it is possibly a mirror of the prodigality of nature which produces a million species of different insects."

All mathematicians recognize the situation that Ulam describes. Only within the narrow perspective of a particular specialty can one see a coherent pattern of development. What are the leading problems? What are the most important recent developments? It is possible to answer such questions within a narrow specialty such as, for example, "nonlinear second-order elliptic partial differential equations."

But to ask the same question in a broader context is almost useless, for two distinct reasons. First of all, there will rarely be any single person who is in command of recent work in more than two or three areas. An overall evaluation demands a synthesis of the judgments of many different people and some will be more critical, some more sympathetic. But even if this difficulty were not present, even if we had judges who knew and understood current research in all of mathematics, we would encounter a second difficulty: we have no stated criteria that would permit us to evaluate work in widely separated fields of mathematics. Consider, say, the two fields of nonlinear wave propagation and category-theoretic logic. From the viewpoint of those working in each of these areas, discoveries of great importance are being made. But it is doubtful if any one person knows what is going on in both of these fields. Certainly ninety-five percent of all professional mathematicians understand neither one nor the other.

Under these conditions, accurate judgment and rational planning are hardly possible. And, in fact, no one attempts to decide (in a global sense, inclusive of all mathematics) what is important, what is ephemeral.

Richard Courant wrote, many years ago, that the river of mathematics, if separated from physics, might break up into many separate little rivulets and finally dry up altogether. What has happened is rather different. It is as if the various streams of mathematics have overflowed their banks, run together, and flooded a vast plain, so that we see countless currents, separating and merging, some of them quite shallow and aimless. Those channels that are still deep and swift-flowing are easy to lose in the general chaos.

Spokesmen for federal funding agencies are very explicit in denying any attempt to evaluate or choose between one area of mathematics and another. If more research proposals are made in area x and are favorably refereed, then more will be funded. In the absence of anyone who feels he has the right or the qualification to make value judgments, decisions are made "by the market" or "by public opinion." But democratic decision-making is supposed to be carried out with controversy and debate to create an informed electorate. In mathematical value judgments, however, we have virtually no debate or discussion, and the vote is more like the economic vote of the consumer who decides to buy or not to buy some commodity. Perhaps classical market economics and modern merchandising theory could shed some light on what will happen. There is no assurance of survival of the fittest, except in the tautological sense that whatever does in fact survive has thereby proved itself fittest—by definition!

Can we try to establish some rational principles by which one could sort through 200,000 theorems a year? Or should we simply accept that there is no more need to choose among theorems than to choose among species of insects? Neither course is entirely satisfactory. Nonetheless decisions are made every day as to what should be published and what should be funded. No one outside the profession is competent to make these decisions; within the profession, almost no one is competent to make them in any context broader than a narrow specialty. There are some exceptional mthematicians whose range of expertise includes several major specialties (for example, probability, combinatorics, and linear operator theory). By forming a committee of such people as these, one can constitute an editorial board for a major journal, or an advisory panel for a federal funding agency. How does such a committee reach its decision? Certainly not by debating and agreeing on fundamental choices of what is most valuable and important in mathematics today.

We find that our judgment of what is valuable in mathematics is based on our notion of the nature and purpose of mathematics itself. What is it to know something in mathematics? What sort of meaning is conveyed by mathematical statements? Thus, unavoidable problems of daily mathematical practice lead to fundamental questions of epistemology and ontology, but most professionals have learned to bypass such questions as irrelevant.

In practice, each member of the panel has a vital commitment to his own area (however skeptical he may be about everybody else) and the committee follows the political principle of nonaggression, or mutual indifference. Each "area" or "field" gets its quota, no one has to justify his own field's existence, and everyone tolerates the continued existence of various other "superfluous" branches of mathematics.

Further Readings. See Bibliography

B. Boos and M. Niss; S. Ulam; Anon. "Federal Funds . . ."

How Much Mathematics Can There Be?

WITH BILLIONS OF BITS of information being processed every second by machine, and with 200,000 mathematical theorems of the traditional, hand-crafted variety produced annually, it is clear that the world is in a Golden Age of mathematical production. Whether it is also a golden age for new mathematical ideas is another question altogether.

It would appear from the record that mankind can go on and on generating mathematics. But this may be a naive assessment based on linear (or exponential) extrapolation, an assessment that fails to take into account diminution due to irrelevance or obsolescence. Nor does it take into account the possibility of internal saturation. And it certainly postulates continuing support from the community at large.

The possibility of internal saturation is intriguing. The argument is that within a fairly limited mode of expression or operation there are only a very limited number of recognizably different forms, and while it would be possible to proliferate these forms indefinitely, a few prototypes adequately express the character of the mode. Thus, although it is said that no two snowflakes are identical, it is generally acknowledged that from the point of view of visual enjoyment, when you have seen a few, you have seen them all.

In mathematics, many areas show signs of internal exhaustion—for example, the elementary geometry of the circle and the triangle, or the classical theory of functions of a complex variable. While one can call on the former to provide five-finger exercises for beginners and the latter for applications to other areas, it seems unlikely that either will ever again produce anything that is both new and startling within its bounded confines.

It seems certain that there is a limit to the amount of living mathematics that humanity can sustain at any time. As new mathematical specialties arise, old ones will have to be neglected.

All experience so far seems to show that there are two inexhaustible sources of new mathematical questions. One source is the development of science and technology, which make ever new demands on mathematics for assistance. The other source is mathematics itself. As it becomes more elaborate and complex, each new, completed result becomes the potential starting point for several new investigations. Each pair of seemingly unrelated mathematical specialties pose an implicit challenge: to find a fruitful connection between them.

Although each special field in mathematics can be expected to become exhausted, and although the exponential growth in mathematical production is bound to level off sooner or later, it is hard to foresee an end to all mathematical production, except as part of an end to mankind's general striving for more knowledge and more power. Such an end to striving may indeed come about some day. Whether this end would be a triumph or a tragedy, it is far beyond any horizon now visible.

Further Readings. See Bibliography

C. S. Fisher; J. von Neumann