CHAPTER 5

The MAGIC FORMULA

Algebra is familiar to most people in the form of equations that must be solved, either equations set as exercises at school or equations formed to model problems in economics, science or some other discipline.

The representation of unknown quantities by symbols, which is fundamental to algebra, evolved slowly. Although Ancient Egyptian and Sumerian mathematicians dealt with problems that involved unknown quantities, they did not express them in the form of equations as we do now. Indeed, not until the late 16th century did the familiar form of an equation evolve. We now have many ways of solving equations, including the use of graphs. This has been made possible by the crowning achievement of René Descartes, who brought together geometry and algebra in the system of Cartesian coordinates which allows an equation to be plotted as a graph.

At this angle, the Tower of Babel defies the rules of God and geometry.

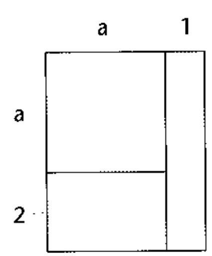
Algebra in the ancient world

It is impossible to disentangle simple algebra from geometry, for it was in problems relating to two- and three-dimensional geometry that algebraic questions first surfaced. Early on, specific, practical problems in algebra were neither systematized nor represented in a way which we would now recognize as algebra – yet they provide the origins of algebra as it was later formulated.

FIELDS AND CELLARS

Babylonian clay tablets in the British Museum include a number of problems which would now be formulated as quadratic or cubic equations. These relate to building projects and involve working with areas and volumes.

Some problems related to dividing up an area in parts with different proportions. It is easy to see how a problem in area can lead to a quadratic equation.



Here the area of the larger (enclosing) rectangle is

$$(a + 2)(a + 1) = a^2 + 3a + 2$$

Similarly, cubic equations can be derived from Babylonian problems relating to digging cellars. The earliest known attempt to write and tackle cubic equations is in the form of 36 problems about construction in a clay tablet nearly 4,000 years old. Such problems were expressed in words by both the Babylonians and Egyptians, and by mathematicians for many centuries afterwards – for example 'the length of a room is the same as its width plus 1 cubit; its height is the same as its length less 1 cubit.'

The method for solving simultaneous equations named after Carl Friedrich Gauss had been used in the East 2,000 years earlier.

The Babylonians did not attempt any general rules or methods of treatment for problems of these types. They dealt only with the specifics of each problem and seem to have had no grasp of a general algorithm that could help them solve all problems of a similar type. The Ancient Egyptians, too, solved practical problems that would now be expressed as linear or quadratic equations, but again without recourse to any formal notation and without recognizing them as equations.

The Chinese text *The Nine Chapters* (2nd to 1st century BC) includes a chapter on solving simultaneous linear equations for two to seven unknowns. They were solved using a counting board or surface and could include negative coefficients. The description of equations with negative coefficients is the earliest known use of negative numbers. The method used is now known in the West as Gaussian elimination after Carl Friedrich Gauss who used it 2,000 years later.

FROM GEOMETRY TOWARDS ALGEBRA

In the middle of the 3rd century AD, the Hellenistic mathematician Diophantus of Alexandria developed new methods for solving problems that would now be shown as linear and quadratic equations. His work, *Arithmetica* (of which only part has survived), contains a number of algebraic equations and methods for solving them. Diophantus applied his methods to the problems in hand, but did not extend them to general solutions. Like the earlier Greeks, he dismissed any solutions that were less than zero, and when an equation yielded more than one solution he stopped after arriving at the first – even if there were an infinite number of solutions (as for an equation of the type $x - y = 3$).

He developed a method for representing equations which was less cumbersome than writing them out in words, but was still not comparable with modern methods. As the Greeks used the letters of their alphabet for numbers, there were no recognizable symbols immediately available to represent variables. We can use x, y, a, b, m, n and so on to stand for variables and constants because we have separate symbols for numbers, and so an expression such as $2x$ is unambiguous. Diophantus adopted some variants on Greek letters, and used symbols to indicate squaring and cubing. His system of abbreviations was an intermediary stage between the purely discursive explanation of problems and the purely symbolic in use now. It also gave him the opportunity, not seen or exploited before, of dealing in higher powers than cubes. Some of his problems include a notation that means 'square-square' or 'cube-cube', indicating powers of 4 and 9 respectively.

In addition, Diophantus had no concept of an equality – of two balanced expressions between which parts could be moved or on which identical operations could be carried out. Nor did Diophantus deal with more than one unknown at a time. He always sought a way to convert a second unknown into an expression built around the first. So, for example, in a problem that calls for two numbers whose

INDIAN QUADRATICS

An ancient Indian text, one of the Sulba sutras written by Baudhayana around the 8th century BC, first cites and then solves quadratic equations of the form $ax^2 = c$ and $ax^2 + bx = c$. These occurred in the context of building altars, and so relate to a practical problem in three dimensions.

sum is 20 and the sum of whose squares is 208, Diophantus would not write, as we may, $x + y = 20$; $x^2 + y^2 = 208$, but might term them $(x + 10)$ and $(x - 10)$, the second equation then becoming $(x + 10)^2 + (x - 10)^2 = 208$.

Diophantine equations

Diophantine equations are those in which all the numbers involved, including those in the solutions, are whole numbers (which can be positive or negative). They fall into three categories: those with no solution, those with a fixed number of solutions and those with infinitely many solutions.

For example, the equation

$$2x + 2y = 1$$

has no solutions, because there are no values for x and y that are whole numbers that can give the answer 1 (the sum of two even numbers is always even).

The equation $x - y = 7$ has infinitely many solutions as we can continue to pick larger and larger values of x and y.

The equation $4x = 8$ has only one solution: $x = 2$.

Diophantine equations are useful for dealing with quantities of objects that cannot be divided – such as numbers of people. So, for instance, if there is a choice of cars to take 24 people on a trip, some of which carry four and some of which carry six passengers, and all must be full, we could write a Diophantine equation, since the only useful solutions assign whole numbers of people to whole numbers of cars:

$$4x + 6y = 24$$

(This has the additional requirement that the values of x and y must both be positive.) Maths problems of the following type use Diophantine equations: 'a boy has spent 96 cents on sweets and bought 4 chocolate mice, 2 lollipops and a chocolate bar. What is the cost of each item?'

Diophantine equations of the form

$$ax + by = c$$

are linear equations (a graph drawn of the equation would be a straight line). Another Diophantine equation,

$$x^2 + y^2 = z^2$$

ORDERS OF EQUATION

Polynomial equations are those that contain a series of terms, each of which has a variable raised to any power, multiplied by a constant (ordinary number). For example in the following equation

$$x^2 + 2x - 8 = 0$$

the first term consists of $x^2 \times 1$, the second of $x^1 \times 2$ and the last the constant 8 (or $x^0 \times -8$). Mathematicians refer to polynomial equations as being of the first order, second order and so on depending on the highest power they contain.

So a quadratic equation such as that above is called a second-order equation; an equation including a cubed term (x^3) is a third-order equation.

relates to Pythagoras' Theorem and produces Pythagorean triplets (e.g., 3, 4, 5: 9 + 16 = 25).

Although Diophantine equations are named after Diophantus, he was not the first to work on them. The Indian Sulba sutras deal with several Diophantine equations. However, Diophantus differed markedly from earlier Indian and Babylonian mathematicians in that his problems were purely theoretical – he was not concerned with building altars, digging cellars or taxing grain, and his numbers do not relate to quantities in the real world. He was also concerned only with precise answers using whole numbers. It is probably for this last reason that there are few cubic equations in Diophantus' *Arithmetica*. Although the questions that Diophantus deals with may not look unusually difficult, his approach was genuinely innovative and has had a lasting effect on later mathematicians. Indeed, it was while trying to generalize a problem raised by Diophantus, to divide a square into two squares, that Fermat arrived at his famous Last Theorem (see page 140).

Going beyond the cube

While Diophantus had a form of notation for powers greater than three he did not make any great use of it. Another Alexandrine, Pappus of Alexandria, approached the issue, but again did not come to grips with it. He was the first to state clearly that linear, or first-order, algebraic problems relate to a single line or one dimension; second-order problems relate to two dimensions or areas, so are planar, and third-order problems relate to three dimensions or volumes, so are solid. Investigating the properties of curves defined by lines in planes and volumes, he came up against the possibility of equations of a higher order. However, he dismissed it since 'there is not anything contained by more than three dimensions'. Diophantus was too much wedded to algebra and Pappus to geometry for either of them to make the conceptual leap into algebraic geometry, though they both approached the jumping-off point. It was one of Pappus' geometric problems of lines and loci that eventually led Descartes to invent algebraic geometry in the 17th century.

OBSERVATIO DOMINI PETRI DE FERMAT

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

Translation of Fermat's Last Theorem: *It is impossible for a cube to be the sum of two cubes, a fourth power to be the sum of two fourth powers, or in general for any number that is a power greater than the second to be the sum of two like powers. I have discovered a truly marvellous demonstration of this proposition that this margin is too narrow to contain.*

AL-MAMUN'S DREAM

The caliph al-Mamun (786–833) is said to have had a dream in which Aristotle appeared to him. As a consequence, the caliph ordered translations to be made of all the Greek texts that could be found. The Arabs had an uneasy peace with the Byzantine empire and negotiated the acquisition of texts through a series of treaties. Under al-Mamun's caliphate and at his House of Wisdom, complete versions of Euclid's *Elements* and Ptolemy's *Almagest* were translated, among others.

A 15th-century painting of Aristotle. Al-Mamun's reign was noted for his huge efforts in the translation of Greek philosophy and science.

The birth of algebra

With the development of the Indo-Arabic number system and the adoption of zero, something approaching modern algebra became possible. The Arab mathematicians, in drawing together the best of Indian and Greek mathematics and extending it, laid the foundations of a proper algebraic system and even gave us the term 'algebra'. They found algebra more appealing than the Greeks had done and there were also spurs to its development within their own society. The incredibly complex laws of inheritance, for example, made the calculation of proportions and fractions a tedious necessity. On top of that, the constant need to find the direction of Mecca made algebra, like geometry, a tool worth developing.

AL JABR WA-L-MUQABALA

The word 'algebra' is derived from the title of a treatise written by the Persian mathematician and member of the House of Wisdom, Muhammad ibn Musa al-Khwarizmi, called *Al-Kitab al-Jabr wa'l-Muqabala* ('The Compendious Book on Calculation by Completion and Balancing'). This presented systematic methods for solving linear and quadratic equations. The modern word 'algorithm' comes from the name 'al-Khwarizmi', too. In his book he gives methods for solving equations of the types $ax^2 = bx$, $ax^2 = c$, $bx = c$, $ax^2 + bx = c$, $ax^2 + c = bx$, and $bx + c = ax^2$ (in modern notation). Like Diophantus, he only considered whole numbers in equations and their solutions; he had the additional requirement that the numbers must also be

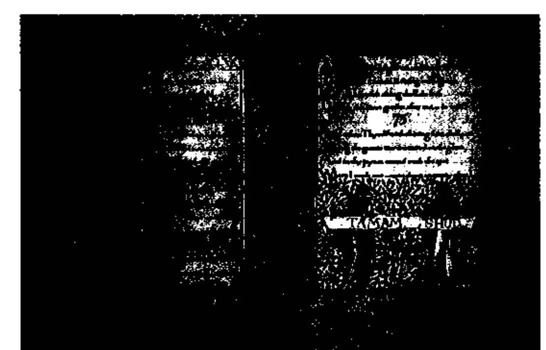
Omar Khayyam was also responsible for the reform of the Persian calendar. His Jalali calendar is the basis of that still in use today in Iran and Afghanistan.

positive, while Diophantus allowed negative numbers. Al-Khwarizmi wrote out all problems and solutions in words and had no symbolic notation. Ironically, since his work is credited with introducing Hindu-Arabic numerals to Europe, he even wrote the numbers out in full.

After showing how to tackle equations, al-Khwarizmi went on to use Euclid's work to provide demonstrations using geometry. Euclid's propositions were entirely geometric, and al-Khwarizmi was the first to apply them to quadratic equations. The method he developed, of systematizing the

GHIYAS AD-DIN ABU AL-FATH OMAR IBN IBRAHIM KHAYYAM NISHABURI (1048–1131)

Omar Khayyam was a mathematician, astronomer and poet born in Iran, probably to a family of tent-makers. He lived most of his life on a modest pension provided by a friend who became grand vizier to the Seljukid empire. His *Treatise on Demonstration of Problems of Algebra* (1070) set out the basic principles of algebra and was responsible for the transmission of the Arab work on algebra to Europe. He worked on the triangular arrangement of numbers known as Pascal's triangle and is sometimes considered the originator of algebraic geometry, which uses geometry to find solutions to algebraic equations.



A 19th-century English translation of Omar Khayyam's collection of four-line poems, the Rubayat. *Many Persian scholars were also poets.*

$$(a+b)^n = a^n + \frac{na^{n-1}b}{1} + \frac{n(n-1)a^{n-2}b^2}{1 \times 2} + \frac{n(n-1)(n-2)a^{n-3}b^3}{1 \times 2 \times 3} + \frac{n(n-1)(n-2)(n-3)a^{n-4}b^4}{1 \times 2 \times 3 \times 4} + \ \dots \ + \frac{nab^{n-1}}{1} + b^n$$

The equation shows how to find the coefficients and variables for any expanded binomial expression of the form $(a + b)^n$.

cases and then applying a geometrical solution, was adopted by later Arab mathematicians and perfected by Omar Khayyam (see below). Al-Khwarizmi's work stands for algebra as Euclid's *Elements* did for geometry, and remained the clearest and best elementary treatment until modern times.

Omar Khayyam followed a similar procedure to al-Khwarizmi, using Greek geometric work on conic sections to demonstrate his solutions to cubic (third-order) equations. Omar Khayyam produced general solutions for cubic equations where the Indian mathematicians had worked only with specific cases. In 13th-century China, Zhu Shijie developed solutions for cubic equations without reference to Omar Khayyam's work.

SHAPES, NUMBERS AND EQUATIONS

In Pascal's triangle, each number is the sum of the two numbers above it. The pattern forms the binomial coefficient series. In Iran, it is called Khayyam's triangle and in China Yang Hui's triangle after the Chinese mathematician Yang Hui (1238–98) who also worked on it.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Before Omar Khayyam wrote on Pascal's triangle, it had been studied in India by Pingala (5th–3rd century BC), though only fragments of his work survive in a later commentary. Another Arab mathematician, Abu Bakr ibn Muhammad ibn al Husayn al-Karaji (*c.*953–1029), had also worked on it and is credited with being the first to derive the binomial theorem (see above):

The Indian mathematician Bhattotpala (*c.*1068) wrote out the triangle up to row 16.

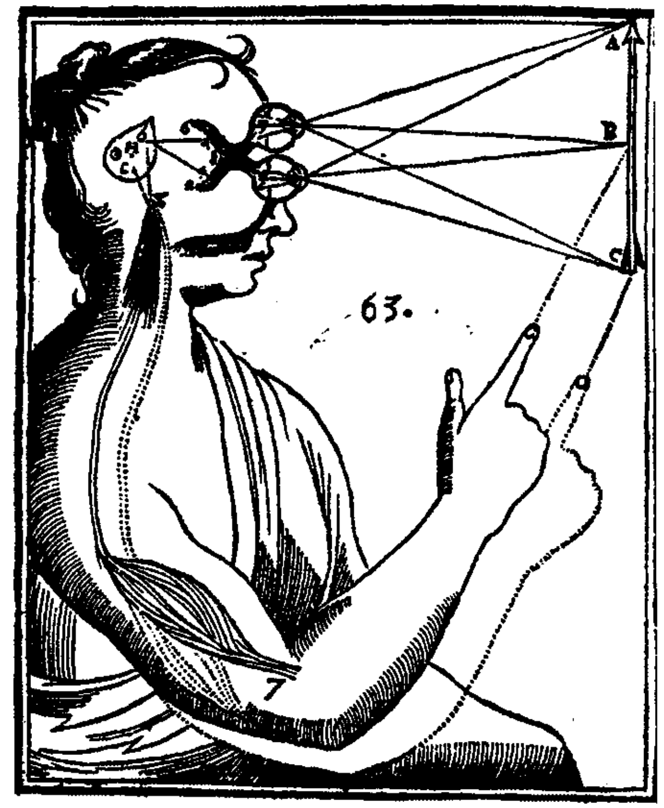
The triangle provides a quick way of expanding expressions such as $(x + y)^3$, since all that is needed is to take the coefficients from (in this case) line 3 (since it is a third-order equation), giving the result:

$$1x^3 + 3x^2y + 3xy^2 + 1y^3.$$

MOVING AWAY FROM AREAS

Although geometry provided good methods of proving algebraic solutions, it was as algebra moved away from the restrictions of real-world geometry that the idea of an abstract equation, relating to numbers rather than measures or quantities, became

> *'Whoever thinks algebra is a trick in obtaining unknowns has thought it in vain. No attention should be paid to the fact that algebra and geometry are different in appearance. Algebras are geometric facts which are proved.'*
>
> Omar Khayyam

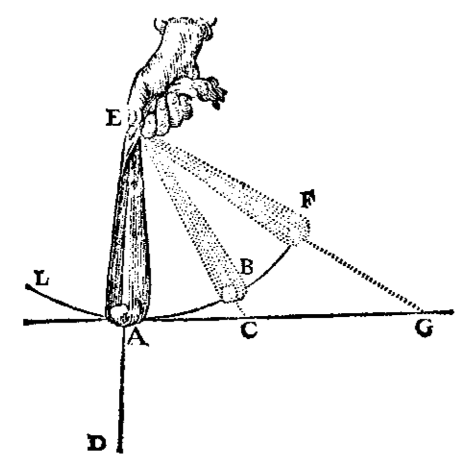


An illustration from Descartes' The World, *in which he set down his theories on light, the senses, biology and many other topics.*

possible. The Arab mathematicians were willing to treat commensurable and incommensurable numbers alongside one another, and to mix magnitudes in different dimensions, both of which the Greeks were unwilling to do.

Combined with the Hindu-Arabic number system and the acceptance of zero, this allowed algebra to move forwards and away from its roots in practical geometry. When Omar Khayyam and al-Khwarizmi had recourse to geometry to demonstrate their algebraic results they were not imagining their algebraic problems in terms of lengths, areas and volumes but using geometry theoretically as a tool to represent algebraic problems.

This relationship between the two, developed over the next 500 years, resulted eventually in the analytic geometry of Descartes and Fermat.



An illustration showing the movement of objects, from Descartes' Principles of Philosophy.

Writing equations

Omar Khayyam died in 1131 and already Arab mathematics was in decline. Scholars from the Arab world were to make few further contributions in the field. Luckily, at the same time that political and religious groups were fracturing the Arab cultural world, the intellectual spirit was reawakening in Europe. During the 12th century Gerard of Cremona translated 87 works of Greek and Arab scholarship into Latin, working at Toledo. These included Ptolemy's *Almagest*, Euclid's *Elements* and al-Khwarizmi's *Algebra*. In England, Robert of Chester translated al-Khwarizmi in 1145 and Adelard of Bath translated Euclid's *Elements* in 1142.

After centuries spent recovering and consolidating earlier learning, European mathematicians began to make their own contribution to the development of algebra. Germany was the focus of these new developments in the 16th century. Perhaps the most important of the new German works on algebra was *Arithmetica integra* by Michael Stifel (*c.*1487–1567). He allowed the use of negative coefficients in quadratic equations and as a consequence reduced the various types of quadratic to a single form. He introduced negative powers, too, giving

$$2^{-1} = 1/2^{1} = {}^{1}/_{2},\ 2^{-2} = 1/2^{2} = {}^{1}/_{4}$$

and so on. Even so, he did not allow negative roots in equations and referred to negative numbers as *numeri absurdi*. He was similarly distrustful of irrational numbers, which he said are 'hidden under some sort of cloud of infinitude'. He proposed using a single letter to denote an unknown quantity, repeating the letter for powers of the number – so if c is the unknown, cc is c^2 and ccc is c^3.

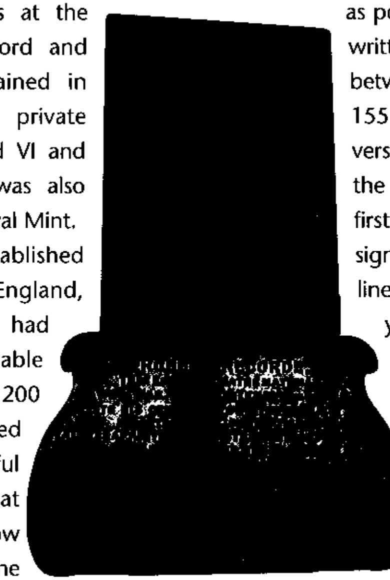
Towards a notation for equations

Algebra without the symbols we use now was cumbersome and long-winded. Yet the modern notation is a late arrival on the scene. In Italy, the symbols $\bar{p}$ and $\bar{m}$ came to be used for plus and minus as abbreviations for the words *più* (more) and *meno* (less). But Latin was full of abbreviations for words and groups of letters that are written repeatedly and this was not particularly original. The introduction of arithmetical operators – symbols showing the type of computation to carry out – did not begin until the late 15th century.

The first symbols to be used were + and –, though originally they were to show a surplus and a deficit in warehouse quantities. They soon took on their modern role as arithmetic operators. They were first printed in a book by Johan Widmann (born *c.*1460), one of several German mathematicians who published on algebra in the late 15th and early 16th centuries.

Even after the development of symbols, many mathematicians continued to follow the rhetorical model, writing out the problems they were posing and solving as discursive text with little or no recourse to symbolic abbreviation (syncopation). Although Western maths did not have a thorough and consistent symbolic algebra until the 17th century, the western part of the Islamic world used symbolic notation in 14th-century commentaries intended for teaching.

ROBERT RECORDE (1510–58)



Robert Recorde was born in Wales and taught mathematics at the Universities of Oxford and Cambridge. He trained in medicine and was private physician to Edward VI and then Mary I. He was also Controller of the Royal Mint.

Recorde re-established mathematics in England, when the country had not seen a notable mathematician for 200 years. He explained everything in careful detail, in steps that were easy to follow and in English, as he wanted to make mathematics as accessible as possible. Most of his works were written in the form of dialogues between a master and a student. In 1551 he published an abridged version of Euclid's *Elements*, making the text available in English for the first time. He first used the equals sign, though using much longer lines than we do now. It took 100 years before the sign was universally accepted above alternative notations.

In 1558 Recorde was imprisoned for failing to pay £1,000 libel charges. He died in prison the same year.

SYMBOL	DATE	SOURCE
+ (plus) - (minus)	1489	Johan Widmann, Germany, *Rechnung auf allen Kauffmanschaften.*
√ (square root)	1525	Christoff Rudolff, Germany, *Die Coss.*
= (equals)	1557	Robert Recorde, England, *The Whetstone of Witte.*
× (multiply)	1618	William Oughtred, England, in an appendix to Edward Wright's translation of John Napier's *Descriptio.*
a, b, c for known quantities (constants) *x, y, z* for unknown quantities (variables)	1637	René Descartes, France, *Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences.*
÷ (divide)	1659	Johann Rahn (or Rhonius), Germany, *Teutsche Algebra.*

'I will sette as I doe often in woorke use, a paire of paralleles, or Gemowe lines of one lengthe, thus: ==, bicause noe. 2. thynges, can be moare equalle.'

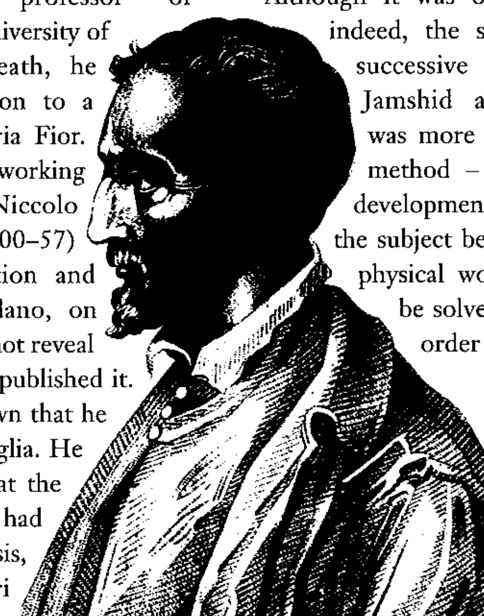
Robert Recorde

THE BEGINNING OF MODERN MATHS?

For all the importance of Stifel's *Arithmetica integra*, it was to be superseded within the year. In 1545 a work appeared that was so revolutionary in its central concept that some people have taken it to mark the start of the modern period in mathematics. In *Ars magna*, Gerolamo Cardano (see panel opposite) explained how to solve cubic (third-order) and even quartic (fourth-order) equations. However, it was not a straightforward triumph of individual genius. The solution to cubics had probably been discovered by Scipione del Ferro (*c.*1465–1526), a professor of mathematics at the University of Bologna. On his death, he passed the information to a student, Antonio Maria Fior. Either from Fior or working independently, Niccolo Tartaglia (*c.*1500–57) discovered the solution and disclosed it to Cardano, on condition that he did not reveal it. Cardano promptly published it. He did, admittedly, own that he had a clue from Tartaglia. He also acknowledged that the solution of the quartic had been by his amanuensis, Ludovico Ferrari (1522–65). Tartaglia, as can be imagined, was not well pleased and the two battled for ten years over Cardano's disclosure. Tartaglia had hoped to retain the revelation of the cubic to publish as the crowning achievement of his career. (Tartaglia had, previously, published the findings of others without acknowledging his own debt, which may reduce our sympathy for him a little.)

Cardano was a little more open-minded with regard to negative numbers than most of his predecessors. Although he just about entertained the possibility of a negative root, he dismissed it as being 'as subtle as it is useless'.

Cardano's book represented the greatest advance in algebra since the Babylonians had discovered how to solve quadratic equations by completing the square. Although it was of no practical use – indeed, the solution of cubics by successive approximation by Jamshid al-Kashi (1380–1429) was more useful than Cardano's method – it stimulated further development of algebra and took the subject beyond the realm of the physical world. If quartics could be solved, then why not fifth-order equations, sixth-order equations and even higher? Suddenly, algebraic problems no longer needed to relate to real-world problems in the dimensions we recognize. Further dimensions, for the sake of mathematics, could be postulated, at least in theory.



It seems likely that Gerolamo Cardano benefited from the work and ideas of other mathematicians in his groundbreaking book Ars magna *(1545).*

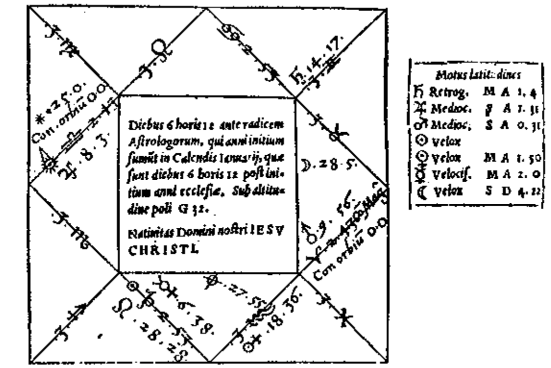
GEROLAMO CARDANO (1501–76)

Born in Pavia, Italy, Gerolamo Cardano was the illegitimate child of Fazio Cardano, a friend of Leonardo da Vinci. His mother tried to abort him, and his three siblings died of plague. After some difficulties in being accepted, he trained as a doctor and was the first to describe typhoid. He became professor of medicine at Pavia in 1543 and at Bologna in 1562.

As well as being a physician, Cardano was one of the foremost mathematicians of his day. His publication of solutions to cubic and quartic equations in *Ars magna* secured his place in history, but he also published the first systematic work on probability a hundred years before Pascal and Fermat.

Cardano's private life was colourful, and certainly fed into his interest in probability. He was always short of money and supplemented his income by gambling and playing chess. His treatment of probability, which he applied to gaming, includes a section on how to cheat effectively.

Life wasn't easy for Cardano. His favourite son was executed in 1560 for poisoning his wife. In 1570, Cardano was accused of heresy and imprisoned for several months for calculating the horoscope of Jesus Christ. As a consequence he lost his professorship at Bologna and the right to publish books. He died on the day he had previously predicted, but he might have aided the fulfilment of his prophecy by committing suicide.



Cardano's horoscope of Jesus Christ that got him into so much trouble. A comet in the ascendant Libra can be seen as an interpretation of the star of Bethlehem, while the star Castor in Gemini predicted violence within Christ's life.

While further dimensions were clearly absurd to Cardano's contemporaries, of interest only in the arena of fantastic mathematical exploration, they would come into their own several centuries later. By opening up the possibility of algebra and algebraic geometry extending into more than three dimensions, Cardano laid the foundations for Riemann geometries and the four-dimensional space-time continuum with which Einstein would remodel the universe (see pages 115–9).

Algebra comes into its own

The golden age of European algebra which began with Cardano's publication of the solution of cubics and quartics, encompassed the legitimization of negative and complex numbers, the development of the Cartesian coordinate system, the marriage of algebra and geometry in analytic geometry as well as considerable steps towards the development of integral calculus.

British mathematicians came into their own again after a long absence from the scene, but did not displace the Italian, German and Polish mathematicians. Some of these men were now writing in their own languages rather than Latin.

TOWARDS COMPLEX NUMBERS

Soon after the Cardano-Tartaglia solution of cubics and quartics appeared, the Italian mathematician Rafael Bombelli (*c.*1526–72) became the first to introduce complex numbers on to the scene. (Complex numbers are those that involve the square root of -1, i.) Working with cube roots, he developed equations which used imaginary roots as a stage in deriving final solutions that are real numbers. He described it as 'a wild thought' and it did not in fact help in his computations, but it did signal the importance that complex numbers were to have for algebra in the future.

DEALING WITH NUMBERS AND NOTATION

Despite all their advances, the algebraists and trigonometers of the 16th century still did not have a widely used notation for decimal fractions. When Rheticus began his most ambitious trigonometric tables, he used triangles with sides of length 10^{15} units to attain the degree of accuracy he wanted without having to use fractions of any kind. (It doesn't matter which units as he didn't actually construct the triangles, just suggested them.)

François Viète (see box opposite) was only a part-time mathematician, but made progress in various fields – arithmetic, trigonometry, geometry and, most importantly, algebra. He was instrumental in bringing about changes in notation that made further progress possible and promoted the use of decimal rather than sexagesimal fractions. Viète's most important contribution was in bringing consistent notation to algebra. This enabled him to develop a systematic way of thinking and a new method of working with general forms of equations. He adopted vowels to represent unknown quantities and consonants to represent known quantities. He also showed how to change the form of equations by multiplying or dividing each side by the same magnitude. For example, he showed how to transform the equation

$$x^3 + ax^2 = b^2x$$

into

$$x^2 + ax = b^2.$$

Viète still did not recognize negative or zero terms, so he could not reduce the number of possible equations to a single form in each order. (We have the form $ax^2 + bx + c = 0$ as the standard form which can describe any quadratic equation because, by allowing a, b

FRANÇOIS VIÈTE (1540–1603)

François Viète was a French mathematician and Huguenot sympathizer. Trained in law, he became a member of the Breton parliament, then of the King's Council serving Henri III and Henri IV. He was proficient at deciphering secret messages intercepted by the French. Indeed, he was so successful that the Spanish accused him of being in league with the devil, complaining to the Pope that the French were using black magic to help them win the war.

Viète made great advances in several fields of mathematics, but always working in his spare time. Being wealthy, he printed numerous of his papers at his own expense. For a period of nearly six years in the second half of the 1580s, he was out of favour at court and concentrated almost exclusively on mathematics. In the 12th century, al-Tusi had found the same method of approximating roots of equations as that discovered by Viète.

or c to be negative or zero, it covers such possibilities as $x^2 - 7 = 0$, where b is 0 and c is negative.)

It is impossible to overstate the importance of good, consistent notation for the progress of algebra. Yet this was not Viète's only achievement. He arrived at formulae for multiple angles, was the first person to use the law of tangents (although he did not publish it) and the first to see that trigonometry could be used to solve cubic equations that could not be reduced. He also produced the first theoretical precise numerical expression for π:

$$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\times\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\times\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\ \ldots$$

Although the method is not new, it was the first time the infinite series had been expressed analytically. Algebra and trigonometry were moving more and more towards a concern with the infinite – both the infinitely large and the infinitely small.

Progress accelerated as a clutch of talented mathematicians applied themselves to developing algebra in its new directions. French mathematician Albert Girard recognized that the number of roots an equation has depends on the order of the equation – so a second-order equation has two roots, a third-order equation has three roots, and so on. The breakthrough came because he was sufficiently open-minded to allow negative and imaginary numbers in roots. Englishman Thomas Harriot (1560–1621) introduced the symbols $>$ and $<$ for greater than and less than. He was also the first proper mathematician to set foot on American soil, having been sent in 1585 by Sir Walter Raleigh as a surveyor. More influential than Viète in promoting the adoption of decimal fractions was the Flemish mathematician Simon Stevin (1548–1620). He also urged the adoption of a decimal system of weights and measures, though this was not to happen for another 200 years. Stevin adopted a notation for powers which is similar to that in use now, using a number in a circle raised above the line to show the power – so 5② means 5^2. He even used fractional powers to show roots, so $5^{1/2}$ means $\sqrt{5}$. But Stevin was primarily a practical mathematician, and he dismissed any consideration of complex numbers.

> *'There are enough legitimate things to work on without the need to get busy on uncertain matter.'*
>
> Simon Stevin, 1585

The confidence with which the best mathematicians now approached algebra – and the distance it had travelled from its roots in real-world geometric problems in up to three dimensions – is clear in the public challenge set in 1593 by the Belgian mathematician Adriaen van Roomen (1561–1615) to solve a 45th-order equation:

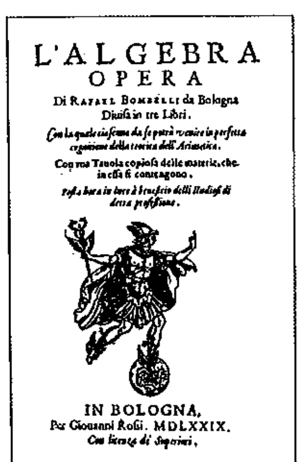
$$x^{45} - 45x^{43} + 945x^{41} - \ldots - 3795x^3 + 45x = K$$

No concept of 45 dimensions was needed. Viète rose to the challenge and solved the equation when an ambassador to the court of Henri IV said that there was no Frenchman capable of it.

THE APPROACH TO ALGEBRAIC GEOMETRY

Viète's solution related to sines and he used his multiple-angle formulae to derive it. In providing a consistent symbolic system for representing algebraic equations, he also made it a matter of choice whether one solved a problem by geometric or algebraic methods. By bringing trigonometry to bear on algebra he was widening the scope of the subject and promoting its alliance with geometry. Viète was in fact one of the first people to view mathematics as a unified whole rather than different branches to be considered separately.

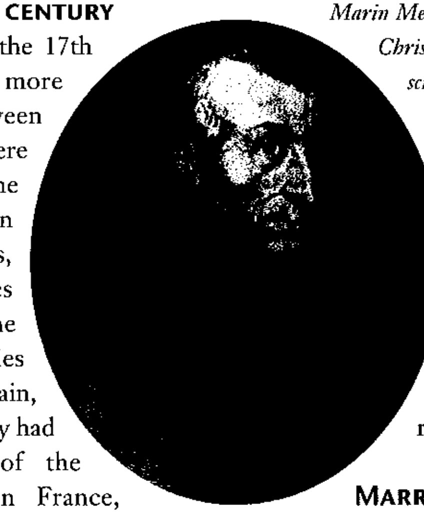
In 1572, Bombelli's *Algebra* had presented many geometric problems which he solved algebraically. For example, he gave algebraic solutions of cubics and then showed geometric demonstrations of his solutions. (However, this part of his treatise was not included in the printed edition and didn't appear until 1929.) Seventy-five years later, Descartes would take geometric problems, convert them to an algebraic form to simplify them as far as possible, then return to geometry for a final solution. In this, his analytic geometry, he completed a journey begun by Apollonius when he showed that conic sections could represent quadratic equations.



The title page to a 1579 edition of Bombelli's Algebra. *The first three volumes of an intended five were published in 1572. Bombelli died that year before he could finalize the last two volumes.*

GIANTS OF THE 17TH CENTURY

From the first half of the 17th century there was more communication between mathematicians than there had been at any time since Plato's Academy. In many countries, mathematical societies grew up alongside the other learned societies then appearing. In Britain, the mathematical society had the enticing name of the 'Invisible College'. In France, communication was further facilitated by Father Marin Mersenne, who corresponded with hundreds of mathematicians, scientists and other learned men, acting as a conduit for knowledge and a sort of early networking guru. This meant that there were fewer incidences of mathematicians privately developing work that was then lost and had no impact on others. Mersenne facilitated disagreement as much as anything, but at least no one was in any doubt about what everyone else was doing. By a process of steady accretion, the foundations of modern mathematics were laid. Two men, both French, were to play a leading role in that process.



Marin Mersenne, who saw it as his Christian duty to disseminate scientific knowledge.

Neither of the two towering figures of the age was a professional mathematician. René Descartes (see page 138) was a minor scion of the French nobility who is more famous as a philosopher than as a mathematician. His explanation of his system of analytic geometry is provided in an appendix to his philosophical text, *Discourse on Method*, as a demonstration of how he used reason to arrive at his results. Pierre de Fermat (see page 139) was a lawyer and then a councillor who pursued his interest in mathematics in his spare time. Yet his ability rivalled that of Descartes.

MARRYING ALGEBRA AND GEOMETRY

Descartes found neither geometry nor algebra entirely satisfactory and set about taking the best of both. By seeing the quantities in his equations as line segments, Descartes avoided any conceptual difficulty in working with higher-order equations and dealing with equations that did not have expressions of the same order on each side. For example, the Greeks could not allow an equation such as $x^2 + bx = a$ because the two parts on the left-hand side are considered areas and that on the right is considered a line; an area and a line cannot be considered equal.

Descartes refined Viète's notation, using letters near the start of the alphabet for known quantities (a, b, c) and letters near the end of the alphabet for unknowns (x, y, z). He used raised numbers to indicate powers and used the symbols for the arithmetical operators which we still use. Only his symbol for equality was different as he had not adopted Robert Recorde's pair of parallel lines.

RENÉ DESCARTES (1596–1650)

René Descartes was born in Tourraine, France. His mother died when he was only a year old. His father remarried and moved away, leaving the infant Descartes in the care of relatives. He trained in law, taking his degree in 1616, and then travelled. It was while he was in Bohemia in 1619 that he developed analytic geometry.

Descartes shared some views and practices of the mystical group the Rosicrucians. Like their full followers, he moved around a good deal, always lived alone and practised medicine without charge, but he rejected their mystical beliefs. He promoted religious tolerance and championed the use of reason in his scientific and philosophical writings.

Descartes has been called the father of modern philosophy for his contention, propounded in his *Discourse on Method,* that knowledge must be acquired through reasoning. He maintained that sensory perceptions are not a reliable guide to the world around us and cannot be depended upon to yield true information. His famous dictum, 'I think, therefore I am', is part of his demonstration of the few things which can be relied upon – the existence of the thinking mind, of God and of the material world. The dichotomy between mind and body was another of his preoccupations. His belief in free will was paramount; he adopted the anti-Calvinist view that salvation can be earned through the operation of free will and does not depend only on God's grace.

Descartes was always sickly and, when Queen Christina of Sweden invited him to her court to teach her philosophy, and demanded that he get up at 5am each day, he quickly succumbed to the Scandinavian winter and died.

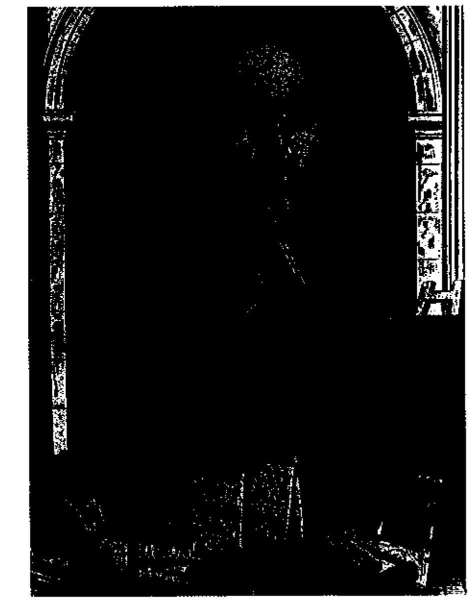
An engraving of Queen Christina of Sweden whose unreasonable demands for instruction by the great philosopher precipitated Descartes' death.

PIERRE DE FERMAT (1601–65)

Born in the Basque region, Fermat studied law and later mathematics. He developed independently of Descartes the principles of using a coordinate system to define the positions of points.

Fermat worked extensively on curves, developing a method for measuring the area under a curve that is similar to integral calculus, and to generalized definitions of common parabolas. He worked extensively, too, on the theory of numbers and corresponded with Blaise Pascal on this subject. This was his only contact with other mathematicians. He was a secretive recluse, who generally communicated only with Marin Mersenne (see page 137).

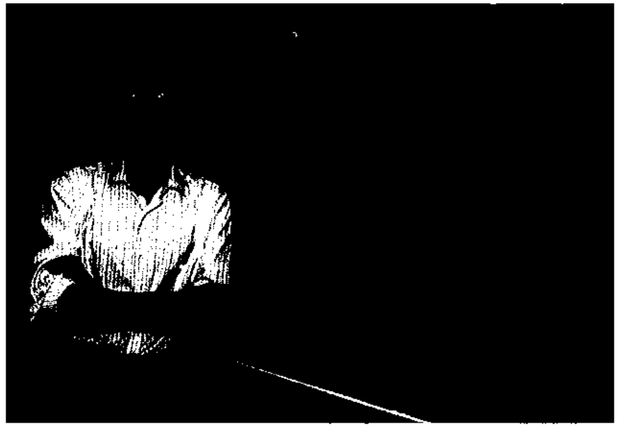
Fermat was the most productive mathematician of his day, but was so reluctant to publish that he gained little credit for his work during his lifetime.



An engraving of Pierre de Fermat taken from Louis Figuier's Vies des Savants Illustres *('Lives of the Great Scientists'), of 1870.*

Descartes proposed that the position of a point in a plane could be identified by reference to two intersecting axes, used as measuring guides, so developing the coordinate system which is now known as the Cartesian system. For all the familiarity of his algebraic notation, Descartes' graphical representations of equations do not all resemble ours, for he never used negative values of x in his graphs. The familiar form of a graph divided into quadrants by axes that cross at (0,0) was introduced later by Isaac Newton. In addition, his axes were not always set at right angles to each other. Descartes believed that any polynomial expression in x and y could be expressed as a curve and studied using analytic geometry.

At the same time as Descartes was formulating his analytic geometry, another Frenchman, Pierre de Fermat, was doing much the same thing. Both arrived at comparable results independently. Fermat stressed that any relationship between x and y defined a curve. He recast Apollonius' work in algebraic terms, aiming to restore some of Apollonius' lost work. Both Descartes and Fermat proposed using a



British mathematician Andrew Wiles, who proved Fermat's Last Theorem, is a professor at Princeton university. He received a knighthood in 2000.

third axis to model three-dimensional curves, but this was not advanced until later in the 17th century.

Neither Descartes nor Fermat sought to publicize their work widely. Descartes did publish his, writing in French so that more people could understand it, but he did not explain in great detail and much of the work was impenetrable to many readers. It is not entirely clear whether Descartes wanted to exclude people whom he felt weren't sufficiently serious or whether he wanted to give his readers the pleasure of discovery by making some of the intellectual leaps and bounds themselves, but either way it did little to help the dissemination of his ideas. Soon, an anonymous introduction was added to his work to help explain it. In 1649 Frans van Schooten published a Latin edition with explanatory commentary.

Fermat was little better at promoting his work than Descartes, being a confirmed recluse who refused to publish. Dissemination of his ideas during his lifetime was almost exclusively through the mediation of Marin Mersenne; indeed, Fermat published only one of his discoveries, the rather obscure rectification of the semicubal parabola (a method for discovering the length of a curved line).

Fermat's Last Theorem

Fermat is most famous now for his so-called 'last' or 'great' theorem. He noted in the margin of his copy of Diophantus' *Arithmetica* that there are no solutions to the equation

$$x^n + y^n = z^n$$

for values of n greater than 2. He added, 'I have discovered a truly marvellous proof of this, which, however, the margin is not large enough to contain' – and so the proof was lost and the subsequent search for it taxed mathematicians for more than 300 years. Because the problem is so easy to understand, many people tried to solve it before it was finally mastered by the English mathematician Andrew Wiles in 1993. Wiles proved Fermat's theorem with a method that uses elliptical curves. He had tried to solve it as a child, as soon as he heard about it, and continued through his degree course in mathematics. Long after he had given up he realized that it was related to his work on curves and returned to the problem again. His proof is highly complex and may well not be the same as that Fermat claimed to have found.

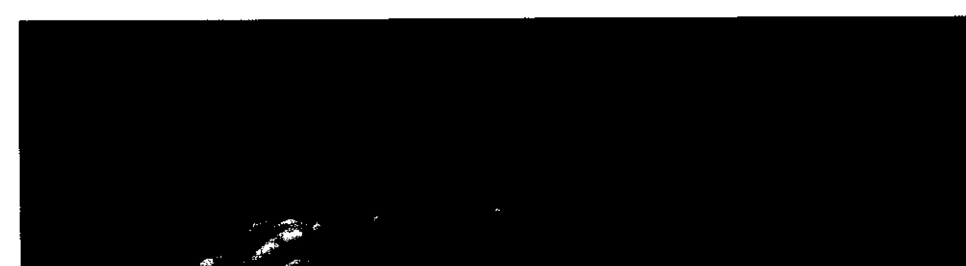
The world is never enough

Descartes brought algebra and geometry together by defining a point by coordinates and using this to draw graphs from equations. In doing this, he provided the means for a later development of algebraic geometry into untold new dimensions.

Any two-dimensional shape can be represented by giving the coordinates of its vertices (corners), each as two numbers. The principle can be extended to three dimensions easily – by giving three coordinates we define a point in three-dimensional space. It is easy to work out the differences between points, too. In a two-dimensional system, with points (a,b) and (c,d) we can use Pythagoras' theorem to work out the distance between the points. We imagine a triangle, with the two points defining the ends of the hypotenuse. The length of this line – the distance between the points – is then $\sqrt{((c - ya)^2 + (d - b)^2)}$. We can extend the same formula to three dimensions: the distance between the points (a,b,c) and (d,e,f) is $\sqrt{((d - a)^2 + (e - b)^2 + (f - c)^2)}$. What is there to stop us taking this further and dealing with distances in four dimensions, defined by four coordinates? Or 26 dimensions? Or 4,519 dimensions? We may have a conceptual objection because we can't visualize four-dimensional space, still less 4,519-dimensional space, but mathematics is not concerned with whether we are comfortable with the concept.

What use is multi-dimensional space? If we can step back from the problems of trying to visualize it as a real-world space, the theoretical space with many dimensions is actually quite useful. We often draw graphs that plot two variables – speed against time, for example, or temperature against growth rate. There are many situations in the real world in which far more than two variables are involved. If we track weather conditions, or the performance of companies in a stock market, or the mortality rates in a population, there are many variables to take into account. By allocating values for perhaps seven, eight or nine variables to each data point we can envisage, if not visualize, a map in seven, eight or nine dimensions from which we can make measurements and predictions could be made. It isn't necessary to draw the map – algebra can take care of the calculations without that – but the conceptual space has been suggested in which the graph exists.

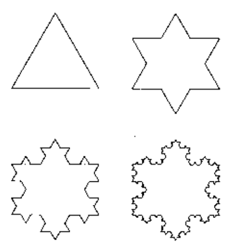
It is very hard for us to visualize space with more than the four dimensions we know to exist, but algebra can work in any number of dimensions.



The Koch snowflake

It is even possible to conceive of geometry in fractional dimensions. A famous model of this is the Koch snowflake, developed by the Swedish mathematician Niels von Koch (1870–1924). The Koch snowflake is an example of a fractal, one of the earliest defined.

Draw an equilateral triangle; divide each side into three equal portions. Remove the middle portion from each side, replacing it with two sides of another equilateral triangle the same size as the removed section. Keep doing this. The result is a shape like a snowflake.

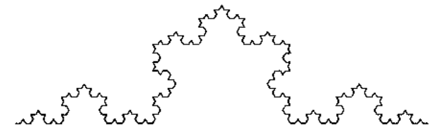


It's possible to carry on doing this an infinite number of times. The result is a shape that has an area defined by the formula

$$\frac{2\sqrt{3}s^2}{5}$$

when s is the measure of one side of the original triangle. However, the perimeter is infinite – an infinite perimeter encloses a finite area.

Carrying out the same operation with a single line segment instead of a triangle, the resulting line approaches a curve as the line segments get smaller and smaller. The curve is called a Koch curve.



The curve has infinite length. The total length increases by one third at each step and so the length after n steps is $(4/3)^n$. It is not a one-dimensional line, as any portion is unmeasurable – it is infinitely long. Yet it is not enclosing an area, so it is not two-dimensional either. It is said to have a fractal dimension of log 4/log 3 ≈ 1.26, greater than the dimension of a line, but less than the dimension of a curve. (A fractal dimension is also called a Hausdorff dimension after one of the founders of modern topology.)

Other fractals

A fractal is a structure in which a pattern is repeated from the large scale to the small scale, so that looking more closely at the structure reveals the same or similar figures. There are many near fractals in nature, including snowflakes, trees, galaxies and blood-vessel networks. Fractals are too irregular to be described using standard Euclidean geometry and generally have a Hausdorff dimension which differs from their normal topological dimension. Fractals are often produced by space-filling algorithms. The Sierpinski triangle is an example. Starting with a simple triangle, make three copies of it at one half the size of the original, and place the copies in the corners of the original. Carry on repeating this step *ad infinitum*. The resulting pattern is identical at any magnification. It was first described by the Polish mathematician Waclaw Sierpiński (1882–1969) in 1915 in the form of a mathematically defined curve rather than a geometric shape. It has a Hausdorff dimension of log 3/log 2 ≈ 1.585.

The best known example of a fractal is the Mandelbrot set, described by the Polish mathematician Benoît Mandelbrot (born 1924). This is the result of drawing a geometric figure of a set of quadratic equations that involve complex numbers.

Mandelbrot drew together earlier examples of fractals, gave them the name 'fractal' and defined their conditions. He explored their prevalence, both in the natural world and in artificial systems such as economics, and determined that they are a very common model, more frequently found than the simple structures of Euclidean geometry. Fractals can often express the 'rough' quality of the real universe, whereas Euclidean geometry deals with smoothness, which is rarely found in nature. Mandelbrot suggested a model of the universe in which stars are fractally distributed. This would solve Olbers' paradox without the need for a Big Bang, though it does not preclude a Big Bang. (Olbers' paradox states that the night sky is dark when it should be bright, since looking in any direction we should see a star. Although it was described by the German astronomer Heinrich Olbers in 1823, it was first noted by Kepler.)

Although fractals generally begin as equations, they are best realized as geometric shapes.

'Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.'

Benoît Mandelbrot

Moving on

With fractals, lines expand into infinity. Working with the graphs produced by Descartes and Fermat, even without the added complexity of infinite line lengths, soon produced a need to calculate the areas under curves and the lengths of curved line segments. The way of dealing with this – and, later, of dealing with fractals – involved looking to the infinitely small. In the late 17th century, mathematicians finally came to grips with the idea of infinity.

Benoît Mandelbrot, 'the father of fractal geometry'.