CHAPTER 4

Elementary Probability Theory

Understandable Statistics, Ninth Edition

Brase & Brase

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CHAPTER 4

Probability

Probability is a numerical measure that indicates the likelihood of an event.

- All probabilities are between 0 and 1, inclusive.
- P = 0 means the event is **impossible**.
- P = 1 means the event is **certain to occur**.
- Events with probabilities near 1 are **likely to occur**.

NOTATION

Events are named with capital letters: A, B, C… **P(A)** = probability of A occurring (read "P of A").

$$0 \leq P(A) \leq 1$$

PROBABILITY ASSIGNMENT METHODS

- **By intuition —** based on experience or judgment.
- **By relative frequency** — $P(A) = f / n$
- **Equally likely outcomes:**

$$P(A) = (\text{Number of Outcomes Favorable to A}) / (\text{Total Number of Outcomes})$$

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Law of Large Numbers

In the long run, as the sample size increases, the relative frequency will get closer and closer to the theoretical probability.

EXAMPLE — PENNY EXPERIMENT

We flip a penny repeatedly. As the number of flips increases, the relative frequency of heads approaches P(head) = 0.50.

Relative Frequency	0.52	0.518	0.495	0.503	0.4996
f = # Heads	104	259	495	1,006	2,498
n = # Flips	200	500	1,000	2,000	5,000

As $n \to \infty$, the relative frequency converges to P(head) = 0.50.

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Probability — Key Definitions

Statistical Experiment: Any random activity that results in a definite outcome.

Event: A collection of one or more outcomes in a statistical experiment.

Simple Event: An event that consists of exactly one outcome in a statistical experiment.

Sample Space (S): The set of all simple events.

Probability Concepts — Review & Key Ideas

Video Reference: *Probability Concepts* — `math.uprag.edu/ProbabilityConcepts.avi`

KEY DISTINCTIONS

Probability
Deals with *populations*.
We know the population and ask: what happens in a sample?

Statistics
Deals with *samples*.
We observe a sample and infer about the unknown population.

ADDITIONAL CONCEPTS

- **Simulation:** Using random processes to model real-world experiments computationally.
- **Subjective Probability:** Probability assigned based on personal judgment, experience, or belief.
- **Relative Frequency Probability:** P(A) = #(A) / #(S)
- **"Hot Hands":** The psychological belief that a player on a winning streak is more likely to continue winning. Statistically debated; most studies show sequential outcomes are independent.

The Sum Rule and The Complement Rule

Sum Rule: The sum of the probabilities of all simple events in the sample space must equal 1.

$$\sum P(\text{all simple events}) = 1$$

Complement: The complement of event A is the event that A does *not* occur, denoted A^c.

$$P(A^c) = 1 - P(A)$$

Probability versus Statistics

Probability

Probability is the field of study that makes statements about what will occur when a sample is drawn from a *known* population.

Statistics

Statistics is the field of study that describes how samples are to be obtained and how inferences are to be made about *unknown* populations.

Independent Events & Multiplication Rules

Independent Events: Two events are independent if the occurrence or nonoccurrence of one does not change the probability of the other.

MULTIPLICATION RULE — INDEPENDENT EVENTS

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

GENERAL MULTIPLICATION RULE (ALL EVENTS)

$$P(A \text{ and } B) = P(A) \cdot P(B|A) \qquad P(A \text{ and } B) = P(B) \cdot P(A|B)$$

CONDITIONAL PROBABILITY (WHEN P(B) ≠ 0)

$$P(A|B) = P(A \text{ and } B) / P(B)$$

Two Fair Dice

Suppose you are going to throw two fair dice. What is the probability of getting a 5 on each die?

SOLUTION

Sample Space: Total outcomes = $6 \times 6 = 36$. Only one outcome is (5, 5).

Multiplication Rule: The two dice are independent.
$P(5 \text{ on die } 1) = 1/6$ | $P(5 \text{ on die } 2) = 1/6$
$P(5 \text{ and } 5) = (1/6)(1/6) =$ **$1/36 \approx 0.0278$**

Drawing Two Aces Without Replacement

Compute the probability of drawing two aces from a well-shuffled deck of 52 cards if the first card is **not replaced** before the second is drawn.

SOLUTION

There are 4 aces in a deck of 52 cards.

P(1st card is an ace) = 4/52
P(2nd card is an ace | 1st was an ace) = 3/51

$P(A_1 \text{ and } A_2) = (4/52)(3/51) = 12/2652 =$ **1/221 ≈ 0.0045**

CHAPTER 4 — VISUAL REFERENCE

Venn Diagrams — Set Operations in Probability

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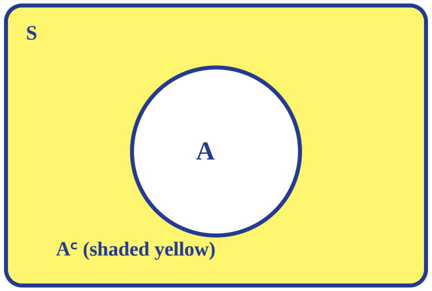
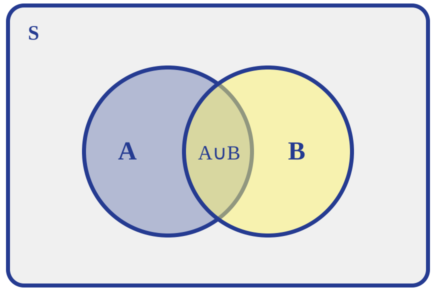


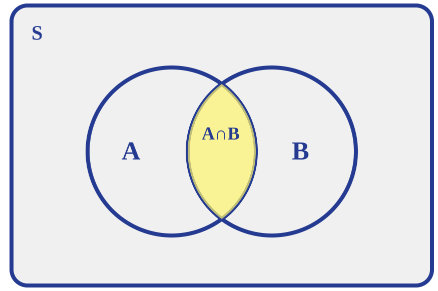
Complement

$P(A^C) = 1 - P(A)$



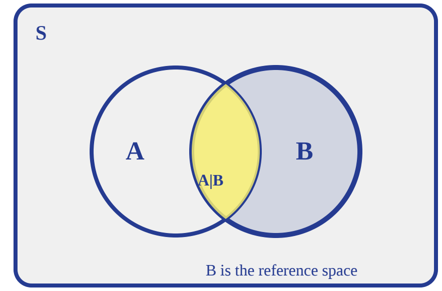
Union (A or B)

$P(A \cup B) = P(A)+P(B)-P(A \cap B)$



Intersection (A and B)

$P(A \cap B) = P(A) \cdot P(B|A)$



Conditional Probability

$P(A|B) = P(A \cap B) / P(B)$

TWO EVENTS OCCURRING TOGETHER

The intersection region represents both events A and B occurring simultaneously: **P(A and B)**.

EITHER OR BOTH OF TWO EVENTS

The union region represents at least one of the events occurring: **P(A or B)**.

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Mutually Exclusive Events

Two events are **mutually exclusive** if they cannot occur at the same time.

Mutually Exclusive = Disjoint

If A and B are mutually exclusive: P(A and B) = 0

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Addition Rules

If A and B are mutually exclusive:

$$P(A \text{ or } B) = P(A) + P(B)$$

If A and B are NOT mutually exclusive:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Why subtract P(A and B)? When A and B can occur together, adding P(A)+P(B) counts the intersection *twice*. We subtract it once to correct for double-counting.

Rosa, Carmen, and Alberto — Exam Problem

Rosa, Carmen and Alberto studying together for an exam. The probability of passing Rosa is 0.65, Carmen pass is 0.75 and Alberto pass is 0.50. The probability that Rosa and Carmen pass is 0.55, that Carmen and Alberto pass is 0.35 and that Rosa and Alberto pass is 0.25. The probability that the three pass is 0.20. What is the probability that: a) At least one of them pass the test? b) Only one of them pass the test? c) Carmen and Alberto pass the exam but not Rosa? d) Alberto not pass the test but at least one of the women? e) None pass the test?

a) $P(R \cup C \cup A)$ = 0.65+0.75+0.50−0.55−0.35−0.25+0.20 = **0.95**

b) Only one: P(R only)=0.65−0.55−0.25+0.20=0.05 | P(C only)=0.75−0.55−0.35+0.20=0.05 | P(A only)=0.50−0.35−0.25+0.20=0.10 → **0.20**

c) $C \cap A \cap R^c$ = $P(C \cap A) - P(R \cap C \cap A)$ = 0.35−0.20 = **0.15**

d) $(R \cup C) \cap A^c$ = $P(R \cup C) - P((R \cup C) \cap A)$ = 0.85−0.40 = **0.45**

e) None: 1−0.95 = **0.05**

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Critical Thinking

Pay attention to translating events described by common English phrases into events described using **and**, **or**, **complement**, or **given**.

Rules and definitions of probabilities have extensive applications in everyday life.

TRANSLATION GUIDE

English Phrase	Probability Operation
"both A and B"	$A \cap B$ (and / intersection)
"at least one of A or B"	$A \cup B$ (or / union)
"A does not occur"	A^c (complement)
"A, knowing that B occurred"	$P(A \mid B)$ (conditional)
"neither A nor B"	$(A \cup B)^c$
"only A, not B"	$A \cap B^c$

Litchfield College of Nursing

At Litchfield College of Nursing, 85% of incoming freshmen nursing students are female and 15% are male. Recent records indicate that 69% of the entering female students will graduate with a BSN degree, while 95% of the male students will obtain a BSN degree. If an incoming freshman nursing student is selected at random, find the following probabilities. (Use 2 decimal places.) (a) P(student will graduate | student is female). (b) P(student will graduate and student is female). (c) P(student will graduate | student is male). (d) P(student will graduate and student is male). (e) P(student will graduate).

SOLUTIONS

Given: $P(F)=0.85$, $P(M)=0.15$, $P(G|F)=0.69$, $P(G|M)=0.95$

a) $P(G|F)$ = **0.69**

b) $P(G \cap F) = 0.69 \times 0.85$ = **0.59**

c) $P(G|M)$ = **0.95**

d) $P(G \cap M) = 0.95 \times 0.15$ = **0.14**

e) $P(G) = 0.59 + 0.14$ = **0.73**

Hospital Problems

PROBLEM 1 — HOSPITAL SMOKERS

70% of hospital patients are women and 20% are smokers. On the other hand 40% of male patients are smokers. Randomly selected a patient from hospital. What is the probability that is a smoker?

$P(W)=0.70$, $P(M)=0.30$, $P(Sm|W)=0.20$, $P(Sm|M)=0.40$
$P(Sm) = 0.20(0.70)+0.40(0.30) = 0.14+0.12 =$ **0.26**

PROBLEM 2 — CESAREAN VS. NATURAL BIRTH

In one hospital, 98% of babies born alive. On the other hand, 40% of all births are by cesarean and 96% of them survive childbirth. If randomly chooses a woman who will not perform caesarean section. What is the probability that the child live?

$P(C)=0.40$, $P(N)=0.60$, $P(S|C)=0.96$, $P(S)=0.98$
$0.98 = 0.96(0.40)+P(S|N)(0.60) \rightarrow P(S|N) = 0.596/0.60 =$ **0.9933 ≈ 0.99**

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Multiplication Rule for Counting & Tree Diagrams

Multiplication Rule for Counting: If event A can occur in *m* ways and event B in *n* ways, then A followed by B can occur in $m \times n$ ways. This rule extends to three, four, or more series of events.

PRACTICE

Outfits: A young man has 4 different pants and 6 different shirts. How many days he is dressed different?

> $4 \times 6 =$ **24 different outfits**

Password (6 characters, 36 options each): A password to access a computer consists of 6 characters, can be letters (26) or numbers (10).
a) Total: $36^6 =$ **2,176,782,336** b) Only numbers: $10^6 =$ **1,000,000** c) At least one letter: $36^6 - 10^6 =$ **2,175,782,336**

TREE DIAGRAMS

Displays outcomes of an experiment as a sequence of branches. Total branches = total outcomes. Each unique outcome = one path from start to finish.

TREE DIAGRAMS WITH PROBABILITY

Label each branch with its probability. Multiply probabilities down a particular branch to get the outcome probability.

Urn with 3 Red & 2 Blue Balls — Without Replacement

Suppose there are five balls in an urn. Three are red and two are blue. We will select a ball, note the color, and, without replacing the first ball, select a second ball. There are four possible outcomes: Red Red, Red Blue, Blue Red, Blue Blue. We can find the probabilities of the outcomes by using the multiplication rule for dependent events.

SOLUTION — DEPENDENT EVENTS

1st Draw	2nd Draw	Outcome	Probability
Red (3/5)	Red (2/4)	R, R	(3/5)(2/4) = 6/20 = **3/10**
Red (3/5)	Blue (2/4)	R, B	(3/5)(2/4) = 6/20 = **3/10**
Blue (2/5)	Red (3/4)	B, R	(2/5)(3/4) = 6/20 = **3/10**
Blue (2/5)	Blue (1/4)	B, B	(2/5)(1/4) = 2/20 = **1/10**

Verification: 3/10 + 3/10 + 3/10 + 1/10 = 10/10 = 1 ✓

Urn with 5 Red & 3 Black — Without Replacement

An urn contains 5 red and 3 black balls. If you drawn two balls, one by one and without replacement. Draw the tree diagram for the sample space.

SOLUTION

1st Draw	2nd Draw	Outcome	Probability
Red (5/8)	Red (4/7)	R, R	(5/8)(4/7) = 20/56 = **5/14**
Red (5/8)	Black (3/7)	R, B	(5/8)(3/7) = **15/56**
Black (3/8)	Red (5/7)	B, R	(3/8)(5/7) = **15/56**
Black (3/8)	Black (2/7)	B, B	(3/8)(2/7) = 6/56 = **3/28**

Verification: 20/56+15/56+15/56+6/56 = 56/56 = 1 ✓

Urn with 5 Red & 3 Black — With Replacement

An urn contains 5 red and 3 black balls. If you drawn two balls, one by one and with replacement. Draw the tree diagram for the sample space.

SOLUTION

1st Draw	2nd Draw	Outcome	Probability
Red (5/8)	Red (5/8)	R, R	(5/8)(5/8) = **25/64**
Red (5/8)	Black (3/8)	R, B	(5/8)(3/8) = **15/64**
Black (3/8)	Red (5/8)	B, R	(3/8)(5/8) = **15/64**
Black (3/8)	Black (3/8)	B, B	(3/8)(3/8) = **9/64**

Verification: 25/64+15/64+15/64+9/64 = 64/64 = 1 ✓

Key Difference: With replacement, the probabilities on the 2nd draw stay the same as the 1st (independent events). Without replacement, they change (dependent events).

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Factorials & Permutations

FACTORIALS

For counting numbers 1, 2, 3, … the symbol **!** is read "factorial".

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

Special cases: $0! = 1$ $1! = 1$

Example: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

PERMUTATIONS

A **permutation** is an ordered grouping of objects.

$${}_nP_r = n! / (n - r)!$$

Olympic Hurdles & Photo Arrangements

EXAMPLE — OLYMPIC HURDLES

Eight athletes compete in the Olympic final of the 110-meter hurdles. Assuming that they cross the line in different moments. How many different ways there to deliver the gold, silver and bronze?

> $_8P_3 = 8!/5! = 8\times7\times6 =$ **336 ways**

EXAMPLE — TEN PEOPLE IN A ROW

Ten people of different heights in a row pose for a photo.

a) How many different pictures can be taken?

> $10! =$ **3,628,800**

b) How many different photographs can be taken if the tallest person and the shortest should not go together in the picture?

> Total: $10! = 3{,}628{,}800$ | Together: $9!\times2! = 725{,}760$ | NOT together: $3{,}628{,}800-725{,}760 =$ **2,903,040**

CHAPTER 4 — PRACTICE

Calculator Exercises & Applications

PART 1 — CALCULATOR EXERCISES

Compute the following. To check, answers are provided.

a) 7! = **5,040** b) $_3P_3$ = 6/1 = **6** c) $_{48}C_3$ = (48×47×46)/6 = **17,296**

PART 2 — DETERMINE METHOD AND SOLVE

a) In how many ways can 5 different cars be parked in a row in a parking lot?

> 5! = **120 ways**

b) In how many different ways can 4 horses be lined up for a race?

> $_4P_4$ = 4! = **24 ways**

c) Suppose 40 cars start at the Indianapolis 500. In how many ways can the top three cars finish the race?

> $_{40}P_3$ = 40×39×38 = **59,280 ways**

Michigan Lotto

Michigan Lotto. The state of Michigan runs a 6-out-of-44-number lotto twice a week that pays at least $1.5 million. You purchase a card for $1 and pick any 6 numbers from 1 to 44. If your 6 numbers match those that the state draws, you win.

i) How many possible 6-number combinations are there for drawing?
${}_{44}C_6 = (44\times43\times42\times41\times40\times39)/720 =$ **7,059,052**

ii) What is the probability of winning the lotto?
$1/7{,}059{,}052 \approx$ **1.417×10^{-7}**

iii) Suppose it takes 10 minutes to pick your numbers and buy a ticket. How many tickets can you buy in 4 days?
$4\times24\times60 = 5{,}760$ min → $5{,}760/10 =$ **576 tickets**

iv) How many people would you have to hire to buy all the tickets and ensure that you win?
$7{,}059{,}052/576 \approx$ **12,255 people**

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Combinations

A **combination** is a grouping that pays no attention to order.

$$_{n}C_{r} = n! / (r! \times (n - r)!)$$

EXAMPLE — COMMITTEE OF 5 FROM 4 WOMEN & 6 MEN

From a group of 4 women and 6 men a committee of 5 members will be chosen.

a) How many committees can be chosen? → $_{10}C_{5}$ = **252**

b) How many committees can choose if we want 3 men? → $_{6}C_{3}\times{}_{4}C_{2} = 20\times6 =$ **120**

c) How many committees can choose if there should be at least one woman? → $252-{}_{6}C_{5} = 252-6 =$ **246**

EXAMPLE — PARTY INVITATIONS (TWO FRIENDS ANGRY)

A woman has 8 friends and you want to invite 5 of them to a party. How many ways can you do if two of them are angry with each other and can not be invited together?

> Total: $_{8}C_{5}=56$ | Both angry: $_{6}C_{3}=20$ | Valid: $56-20 =$ **36 ways**

Probability Problems Using Combinations

EXAMPLE — SCIENTISTS FOR SOUTH AMERICA CONGRESS

A group of scientists of 5 Argentine, 3 Chileans, 2 Colombians and 2 Peruvians will randomly select 6, to represent South America in a world congress. What is the probability that:

a) 2 Argentines and 2 Chileans is elected?

> Total: ${}_{12}C_6=924$ | Favorable: ${}_5C_2 \times {}_3C_2 \times {}_4C_2 = 10\times3\times6 = 180 \rightarrow P = 180/924 \approx$ **0.195**

b) Take at least one elected Peruvian?

> $P(\text{no Peruvian}) = {}_{10}C_6/{}_{12}C_6 = 210/924 \rightarrow P(\geq1) = 1-210/924 = 714/924 \approx$ **0.773**

EXAMPLE — EXAM PROBLEMS

A teacher assigned a week before the test a set of 10 problems. The exam will consist of 5 problems selected at random from among the 10 allotted. A student could only resolve 7 of those problems. What is the probability that student:

a) Answer 3 out of 5 good questions?

> ${}_7C_3 \times {}_3C_2 / {}_{10}C_5 = (35\times3)/252 = 105/252 =$ **5/12 ≈ 0.417**

b) Have at least 4 good questions?

> $P(4)=({}_7C_4 \times {}_3C_1)/252=105/252$ | $P(5)=({}_7C_5 \times {}_3C_0)/252=21/252 \rightarrow P(\geq4) = 126/252 =$ **1/2 = 0.50**